M. Sc (Math) Code No. 16MMC21C3 **Semester-I**

Ordinary Differential Equations

DIRECTORATE OF DISTANCE EDUCATION

MAHARSHI DAYANAND UNIVERSITY, ROHTAK

(A State University established under Haryana Act No. XXV of 1975)

NAAC 'A +’ Grade Accredited University

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Paper Code: 16MMC21C3
Ordinary Differential Equations

M. Marks = 100
Term End Examination = 80
Assignment = 20
Time = 3 Hours

Course Outcomes

Students would be able to:

CO1 Apply differential equations to variety of problems in diversified fields of life.

CO2 Learn use of differential equations for modeling and solving real life problems.

CO3 Interpret the obtained solutions in terms of the physical quantities involved in the original problem under reference.

CO4 Use various methods of approximation to get qualitative information about the general behaviour of the solutions of various problems.

Section - I

Preliminaries, ε-approximate solution, Cauchy-Euler construction of an ε-approximate solution of an initial value problem, Equicontinuous family of functions, Ascoli-Arzela Lemma, Cauchy-Peano existence theorem.

Lipschitz condition, Picards-Lindelof existence and uniqueness theorem for dy/dt =f(t,y), Solution of initial-value problems by Picards method, Dependence of solutions on initial conditions ***(Relevant topics from the books by Coddington& Levinson, and Ross)***.

Section - II

Linear systems, Matrix method for homogeneous first order system of linear differential equations, Fundamental set of solutions, Fundamental matrix of solutions, Wronskian of solutions, Basic theory of the homogeneous linear system, Abel-Liouville formula, Nonhomogeneous linear system.

Strum Theory, Self-adjoint equations of the second order, Abel formula, Strum Separation theorem, Strum Fundamental comparison theorem.

(Relevant topics from chapters 7 and 11 of book by Ross)

Section - III

Nonlinear differential systems, Phase plane, Path, Critical points, Autonomous systems, Isolated critical points, Path approaching a critical point, Path entering a critical point, Types of critical points- Center, Saddle points, Spiral points, Node points, Stability of critical points, Asymptotically stable points, Unstable points, Critical points and paths of linear systems. Almost linear systems. **(Relevant topics from chapter 13 of book by Ross)**.

Section - IV

Nonlinear conservative dynamical system, Dependence on a parameter, Liapunov direct method, Limit cycles, Periodic solutions, Bendixson nonexistence criterion, Poincore-Bendixson theorem(statement only), Index of a critical point.

Strum-Liouville problems, Orthogonality of characteristic functions. **(Relevant topics from**

chapters 12 and 13 of the book by Ross)

Note : The question paper of each course will consist of **five** Sections. Each of the sections **I to IV** will contain **two** questions and the students shall be asked to attempt **one** question from each. **Section-V** shall be **compulsory** and will contain **eight** short answer type questions without any internal choice covering the entire syllabus.

Books Recommended:

1. Coddington, E.A., Levinson, N., Theory of ordinary differential equations, Tata McGraw Hill, 2000.
2. Ross, S.L., Differential equations, John Wiley and Sons Inc., New York, 1984.
3. Boyce, W.E., Diprima, R.C., Elementary differential equations and boundary value problems, John Wiley and Sons, Inc., New York, 4th edition, 1986.
4. Simmon, G.F., Differential Equations, Tata McGraw Hill, New Delhi, 1993.

Contents

1

Solution of Differential Equations

Structure

1.1. Introduction. This chapter contains many important results for obtaining the solutions of given differential equations and then the dependence of solutions on initial conditions is discussed.

1.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Construction of an approximate solution for a given differential equation.

(ii) Existence of solution with the help of sequence of approximate solutions.

(iii) Uniqueness of solution.

1.1.2. Keywords. ε-approximate solution, Lipschitz condition, Successive approximations.

1.2. Differential Equations.

Suppose f is a complex valued function defined for all real t in an interval I and for complex y in some set D. The value of f at (t, y) is denoted by $f(t, y)$. An important problem associated with f is to find a complex-valued function ϕ on I, which is differentiable there, such that for all t on I,

(i) $\left(t, \phi(t)\right) \in D \;\; (t \in I)$

(ii) $\phi'(t) = f\left(t, \phi(t)\right)$

This problem is called finding a solution of an ordinary differential equation of the first order denoted by

$$y' = f(t, y) \qquad (1)$$

The ordinary refers to the fact that only ordinary derivatives enter into the problem and not partial derivatives. If such a function ϕ exists on I satisfying (i) and (ii) there, then ϕ is called a solution of (1) on the interval I.

Notation. The set of all complex-valued functions having k continuous derivatives on an open interval I is denoted by $C^k(I)$.

Remark. Clearly if ϕ is a solution of (1) satisfying (i) and (ii) then $\phi \in C'(I)$ because ϕ' is continuous on interval I on account of condition (ii).

1.2.1. Geometrical Interpretation. In geometrical language $\dfrac{dy}{dx} = f(t, y)$ represents a slope of $f(t, y)$ at each point of D. A solution ϕ on I is a function whose graph has the slope $f\left(t, \phi(t)\right)$ for each $t \in I$.

Note. The problem (1) may have many solutions on an interval I and, therefore, we shall be interesting in finding a solution passing through a given point in (t, y)-plane.

1.2.2. Initial value problem. To find a solution of ordinary differential equation $\dfrac{dy}{dt} = f(t, y)$ on an open interval J satisfying a given condition $\phi(t_0) = y_0$ for some $t_0 \in J$.

1.2.3. Leibnitz rule. If $a(\alpha)$ and $b(\alpha)$ are continuous and differentiable functions of α, then

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)dx = \int_a^b \frac{\partial}{\partial\alpha}\left(f(x, \alpha)\right)dx + f\left(b(\alpha), \alpha\right)\frac{db(\alpha)}{d\alpha} - f\left(a(\alpha), \alpha\right)\frac{da(\alpha)}{d\alpha}.$$

1.2.4. Theorem. A function ϕ is a solution of the initial value problem

$$\frac{dy}{dt} = f(t, y) \text{ satisfying } y(t_0) = y_0 \qquad (1)$$

if and only if it is a solution of the integral equation

$$y = y_0 + \int_{t_0}^{t} f\left(s, y(s)\right)ds. \qquad (2)$$

Proof. Suppose ϕ is a solution of the (1) on open interval J. Then

$$\phi'(t) = f(t, \phi(t)) \qquad (3)$$

on J. Integrating (3) over $[t_0, t]$,

$$\phi(t) - \phi(t_0) = \int_{t_0}^{t} f(s, \phi(s))\, \mathrm{ds}$$

$$\Rightarrow \qquad \phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds.$$

We see that ϕ is a solution of (2).

Conversely, Suppose ϕ is a solution of the integral (2). Then

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))\, \mathrm{ds} \qquad (4)$$

Putting $t = t_0$, we get $\phi(t_0) = y_0$ which is the initial condition of (1).

Now, differentiating (4) both sides w.r.t. 't',

$$\frac{d}{dt}(\phi(t)) = 0 + \frac{d}{dt}\left[\int_{t_0}^{t} f(s, +(s))\, ds\right]$$

$$= \int_{t_0}^{t} \frac{\partial}{\partial t}\left[f(s, \phi(s))\right] ds + f(t, \phi(t))\frac{\partial}{\partial t}(t) - f(t_0, \phi(t_0))\frac{\partial}{\partial t}(t_0)$$

$$\Rightarrow \qquad \phi'(t) = f(t, \phi(t))$$. Thus ϕ is a solution of (1).

Remark.

1. For a given continuous function $f(t, y)$ on a domain D, the first question to be answered is "whether there exist a solution of the ordinary differential equation $\frac{dy}{dt} = f(t, y)$ for all $t \in J$". The answer is yes, if the interval J is properly prescribed.

2. An indication of the limitation of any general existence theorem can be seen by considering the simple example:

$$\frac{dy}{dt} = y^2$$

It is clear that a solution of this equation which passes through the point $(1, -1)$ is given by $\phi(t) = -t^{-1}$. However this solution does not exist at $t = 0$, although $f(t, y) = y^2$ is continuous there. This shows that any general existence theorem will necessarily have to be of local nature, and existence in large can only be asserted under additional conditions on $f(t, y)$.

1.2.5. Local existence. Now our problem is "Does the initial value problem $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$ have a solution $y = y(t)$ defined for t near t_0". For this we shall be constructing a sequence of approximate solutions which tend to a solution of given initial value problem.

1.3. ε-approximate solution. Let f be a real valued continuous function on a domain D in the (t, y) plane. An $\in$-approximate solution of $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$ on a t interval J is function $\phi \in C$ on J such that

(i) $\left(t, \phi(t)\right) \in D$ for all $t \in J$.

(ii) $\phi \in C'(J)$, except possibly for a finite set S of points on J, where ϕ' may have jump discontinuities (means left and right limits exist but are not equal).

(iii) $\left|\phi'(t) - f\left(t, \phi(t)\right)\right| < \in$ for all $t \in J - S$.

Remark.

1. Any function $\phi \in C$ satisfying property (ii) on J is said to have a piece wise continuous derivative on J and this is denoted by $\phi \in C'_p(J)$.

2. Let us consider the rectangle R,

$$R = \left\{(t, y) : |t - t_0| \le a, |y - y_0| \le b, a > 0, b > 0\right\}.$$

If $f \in C(R)$ then f is a continuous function on rectangle R and since R is a closed set so f is bounded there. Let, $M = \max. |f(t, y)|$ on R for all $(t, y) \in R$ and let $\alpha = \min. \left\{a, \frac{b}{M}\right\}$.

Then, $\alpha \le a$ and $\alpha \le \frac{b}{M}$.

1.3.1. Cauchy-Euler Construction of an ε-approximate solution.. Let $f \in C$ on the rectangle R. Given any $\in > 0$, there exists an $\in$-approximate solution ϕ of ordinary differential equation

$$\frac{dy}{dt} = f(t, y) \qquad (1)$$

on an interval $I = \left\{t : |t - t_0| \le \alpha\right\}$ such that $\phi(t_0) = y_0$.

(Note that here $\alpha = \min. \left\{a, \frac{b}{M}\right\}$ and $M = \max |f(t, y)|$ on R.)

Proof. Let $\in > 0$ be given. We shall construct an $\in$-approximate solution for the interval $[t_0, t_0 + \alpha]$. *A* similar construction will define it for the interval $[t_0 - \alpha, t_0]$. This approximate solution will consist of a polygonal path starting at (t_0, y_0), that is, a finite number of straight line segments joined end to end.

Since $f \in C$ on R, it is *u*niformly continuous on R [Because a continuous function on a closed domain is *u*niformly continuous on that domain]. Hence, there exist a real number $\delta_\in > 0$ (for a given $\in > 0$) such that

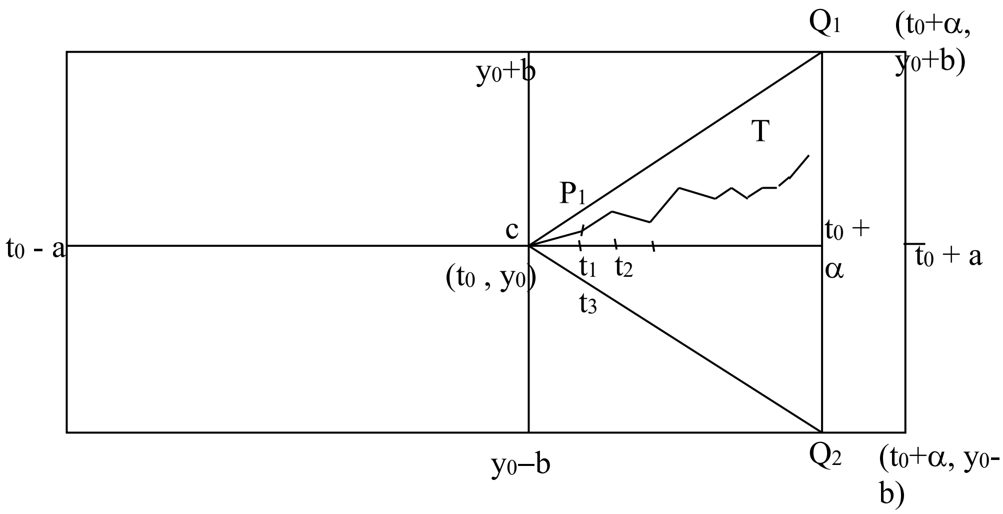
$$\left| f(t, y) - f\left(\bar{t}, \bar{y}\right)\right| \leq \in \tag{2}$$

provided that $\left|t - \bar{t}\right| \leq \delta_\in$ and $\left|y - \bar{y}\right| \leq \delta_\in$ (3)

Now divide the interval $[t_0, t_0 + \alpha]$ into a parts $t = t_0 < t_1 < t_2 < \ldots\ldots < t_n = t_0 + \alpha$ in such a way that

$$\max \left|t_k - t_{k-1}\right| \leq \min.\left(\delta_\in, \frac{\delta_\in}{M}\right) \tag{4}$$

Starting from the point $C(t_0, y_0)$ we construct the straight line segment with slope $f(t_0, y_0)$ proceeding to the right of t_0 until it intersects with the line $t = t_1$ at some point (t_1, y_1) where $y_1 = y(t_1)$ and let this point be P_1.



This line segment CP_1 must lie inside the triangular region bounded by the lines $C\,Q_1$, $C\,Q_2$ and the third line Q_1Q_2. Here, slope of $CQ_1 = \dfrac{y_0 + b - y_0}{t_0 + \alpha - t_0} = \dfrac{b}{\alpha}\;\dfrac{b}{\dfrac{b}{M}} = M$.

Note that α is taken to be $\dfrac{b}{M}$ in figure. Similarly, slope of $CQ_2 = -M$.

Now, at the point (t_1, y_1), we construct to the right of t_1, a straight line segment with slope $f((t_1, y_1)$ till the intersection with $t = t_2$, say at (t_2, y_2). Continuing in this fashion, in a finite number of steps the resultant path ϕ will meet the line $t_0+\alpha$. Further the path will lie completely within the triangular region. This ϕ is the required $\in$-approximate solution. Analytically, it may be expressed as $\phi(t_0)=y_0$

$$\phi(t) = \phi(t_{k-1}) + f(t_{k-1}, \phi(t_{k-1}))\,(t - t_{k-1}) \qquad (5)$$

for $t_{k-1} < t \le t_k$ and $k = 1, 2, \ldots, n$. Note that **(**5) is equation of straight line passing through the point $\left(t_{k-1}, \phi(t_{k-1})\right)$ and slope $f\left(t_{k-1}, \phi(t_{k-1})\right)$. From the construction of ϕ, it is clear that $\phi \in C'_p$ on $[t_0, t_0+\alpha]$ as its derivative may not be continuous at finite number of joins of end points of line-segments. Also for t, $\bar{t} \in [t_0, t_0+\alpha]$, we have

$$\left|\phi(t) - \phi(\bar{t})\right| = \left|f\left(t_{k-1}, \phi(t_{k-1})\right)(t-\bar{t})\right| \le M\left|t-\bar{t}\right| \qquad (6)$$

using (5). If t is such that $t_{k-1} < t < t_k$, then **(**6) together with **(**4) gives

$$\left|\phi(t) - \phi(t_{k-1})\right| \le M\left|t - t_{k-1}\right| \le M.\frac{\delta_\in}{M} = \delta_\in$$

But from (2) and (5), we get

$$\left|\phi'(t) - f\left(t, \phi(t)\right)\right| = \left|f\left(t_{k-1}, \phi(t_{k-1})\right) - f\left(t, \phi(t)\right)\right| \le \in$$

Here we can apply (2) because $\left|t - t_{k-1}\right| \le \delta_\in$ (By (4)) and $\left|\phi(t) - \phi(t_{k-1})\right| \le \delta_\in$ as proved above.

This shows that ϕ is an $\in$-approximate solution as required.

Note. After finding an $\in$-approximate solution, we shall prove in further studies that there exist a sequence of these approximate solutions which tend to a solution.

1.4. Equicontinuous family of functions. *A* family of functions $F = \{f\}$ defined on a real interval J is said to be equicontinuous on J, if given $\in > 0$, there exist a $\delta_\in > 0$, independent of $f \in F$ and also t, $\bar{t} \in J$ such that

$$\left|f(t) - f(\bar{t})\right| < \in \text{ whenever } \left|t-\bar{t}\right| < \delta_\in .$$

1.4.1. Ascoli lemma. On a bounded interval J, let $F = \{f\}$ be an infinite family of *u*niformly bounded and equicontinuous set of functions. Then F contains a sequence $\{f_n\}$, $n = 1, 2, \ldots$ which is *u*niformly convergent on J.

Proof. Let $\{r_k\}$, $k = 1, 2, 3, \ldots$ be the rational numbers in J enumerated in some order. The set of numbers $\{f(r_1)\}$, $f \in F$, is bounded, and hence there exist a sequence of distinct functions $\{f_{n1}\}$, $f_{n1} \in F$, such that the sequence $\{f_{n1}(r_1)\}$ is convergent. Similarly the set of numbers $\{f_{n1}(r_2)\}$ has a convergent subsequence $\{f_{n2}(r_2)\}$.

Continuing in this way, an infinite set of functions $f_{nk} \in F$, where $n, k = 1, 2, \ldots$ is obtained which have the property that $\{f_{nk}\}$ converges at $r_1, r_2, \ldots, r_k$. Define f_n to be the function f_{nn}. Then $< f_n >$ is the required sequence which is *u*niformly convergent on J.

Clearly $< f_n >$ converges at each of the rationals on J. that is, $< f_n(r_k) >$ is convergent for every rational r_k on J. We know that every convergent sequence is a Cauchy sequence, hence for every $\in > 0$, and $r_k \in J$, there exist an integer $N_\in(r_k)$ such that

$$|f_n(r_k) - f_m(r_k)| < \in \text{ for all } n, m > N_\in(r_k). \qquad (1)$$

Now the set $F = \{f\}$ is an equicontinuous set, so for given $\in > 0$, there exist a $\delta_\in$, independent of t and $\bar{t}$ and $f \in F$ such that

$$|f(t) - f(\bar{t})| < \in \text{ whenever } |t - \bar{t}| \leq \delta_\in \qquad (2)$$

Now, divide the interval J into a finite number of subintervals $J_1, J_2, \ldots, J_p$ such that the length of the largest subinterval is less than $\delta_\in$. For each J_k choose a rational number $\bar{r}_k \in J_k$. Hence

$$|f_n(t) - f_m(t)| \leq |f_n(t) - f_n(\bar{r}_k) + f_n(\bar{r}_k) - f_m(\bar{r}_k) + f_m(\bar{r}_k) - f_m(t)|$$

$$\leq |f_n(t) - f_n(\bar{r}_k)| + |f_n(\bar{r}_k) - f_m(\bar{r}_k)| + |f_m(\bar{r}_k) - f_m(t)|$$

$$< \in + \in + \in = 3\in \qquad \text{[Using (1) and (2)]}$$

provided that $n, m >$ max. $\{N_\in(\bar{r}_1), \ldots, N_\in(\bar{r}_k)\}$. This proves the *u*niform convergence of the sequence $\{f_n\}$ on J in view of Cauchy criteria for *u*niform convergence.

1.4.2. Cauchy-peano existence theorem. If $f \in C$ on the rectangle R, *t*hen thee exist a solution $\phi \in C'$ of the differential equation $\frac{dy}{dt} = f(t, y)$ on the interval $|t - t_0| \leq \alpha$ for which $\phi(t_0) = y_0$ where α = min. $\left(a, \frac{b}{M}\right)$ and $M = \max |f(t, y)|$, $(t, y) \in R$.

Proof. Let $\{\in_n\}$, $n = 1, 2, \ldots$ be a montonically decreasing sequence of positive real numbers tending to zero as $n \to \infty$. By the theorem of $\in$- approximate solutions, for each $\in_n$ there exist an $\in_n$-approximate solution, say, ϕ_n, of the differential equation $\frac{dy}{dt} = f(t, y)$ on the interval $|t - t_0| \leq \alpha$ such that $\phi_n(t_0) = y_0$ Here, we know that $R = \{(t, y)$. $|t - t_0| \leq a$, $|y - y_0| \leq b$, $a > 0$, $b > 0\}$ and $M = \max|f(t, y)|$ for $(t, y) \in R$ and α = min. $\left(a, \frac{b}{M}\right)$

Now, by the theorem of $\in$-approximate solution, we know that

$$\left|\phi_n(t)-\phi_n(\bar{t})\right| \le M\left|t-\bar{t}\right| \text{ for } t,\ \bar{t} \in [t_0, t_0+\alpha] \qquad (1)$$

Applying (1) to $\bar{t} = t_0$ and since we know that $|t-t_0| \le \alpha = \dfrac{b}{M}$, it follows that $|\phi_n(t)-\phi_n(t_0)| \le b$

$$\Rightarrow \qquad |\phi_n(t)-y_0| \le b$$

Now, $|\phi_n(t)| = |\phi_n(t)-y_0+y_0| \le |\phi_n(t)-y_0|+|y_0| \le b+|y_0|$.

This shows that the sequence $\{\phi_n(t)\}$ is *u*niformly bounded by $b+|y_0|$. Moreover, (1) implies that the sequence $\{\phi_n\}$ is an equicontinuous set. Therefore by the Ascoli lemma, there exist a subsequence $\{\phi_{nk}\}$, $k = 1, 2,\ldots$, of $\{\phi_n\}$, converging *u*niformly on $[t_0-\alpha, t_0+\alpha]$ to a limit function ϕ which must be continuous since each ϕ_{nk} is continuous. Now we shall show that this limit function ϕ is a solution of given ordinary differential equation which gives the required specifications. To show this, we write the relation defining ϕ_n as an $\in_n$- approximate solution in the integral form, as follows

$$\phi_n(t) = y_0 + \int_{t_0}^{t}\left[f\left(s,\ \phi_n(s)\right)+\Delta_n(s)\right]ds \qquad (2)$$

where $\Delta_n(s) = \phi_n'(s)-f\left(s,\ \phi_n(s)\right)$ at those points where ϕ'_n exists and $\Delta_n(s) = 0$ otherwise . Because ϕ'_n is an $\in_n$-approximate solution therefore

$$|\Delta_n(s)| \le \in_n \qquad (3)$$

Since f is *u*niformly continuous on R and $\phi_{nk} \to \phi$ *u*niformly on the interval $[t_0-\alpha, t_0+\alpha]$ as $k\to\infty$, it follows that $f(t,\phi_{nk}(t)) \to f(t,\phi(t))$ *u*niformly on $[t_0-\alpha, t_0+\alpha]$ as $k\to\infty$.

Replacing n by n_k in (2), letting $k\to\infty$ and *u*sing (3), we obtain

$$\phi(t) = y_0 + \int_{t_0}^{t} f\left(s,\ \phi(s)\right)ds \qquad (4)$$

From this (4), on putting $t = t_0$, we get $\phi(t_0) = y_0 + 0 = y_0$ and on differentiating (4), we get $\phi'(t) = f(t,\phi(t))$. It is clear from this that $\phi(t)$ is a solution of given ordinary differential equation on $|t-t_0| \le \alpha$ and is of class C'.

Remarks.

1. In general, the choice of a subsequence of $\{\phi_n\}$ in the above proof is necessary because there may exist polygonal paths $\{\phi_n\}$ which diverge everywhere on a whole interval about $t = t_0$ as $\in_n \to 0$.

2. If uniqueness of solution is assured then the choice of a subsequence in above theorem is unnecessary.

Proof. If it is assumed that a solution of differential equation $\frac{dy}{dt} = f(t, y)$ through (t_0, y_0) (if it exists) is unique then every sequence of polygonal paths $\{\phi_n\}$ for which $\in_n \to 0$ must converge on $|t - t_0| \leq \alpha$, and hence uniformly, to a solution because $\{\phi_n\}$ is an equicontinuous set on $|t - t_0| \leq \alpha$. Suppose this were false. Then there would exist a sequence of polygonal paths $\{\phi_n\}$ divergent at some point on $|t - t_0| \leq \alpha$. This implies the existence of at least two subsequences of $\{\phi_n\}$ tending to different limit functions. Both will be solutions and this gives a contradiction because it is pre-assumed that solution is unique.

3. It can happen that the choice of a subsequence is unnecessary even though uniqueness is not satisfied. It can be explained by taking an example. Consider the example

$$\frac{dy}{dt} = y^{1/3} \qquad (1)$$

There are an infinite number of solutions starting at (0, 0) which exist on $J = [0, 1]$. For any c such that $0 \leq c \leq 1$, the function ϕ_c defined by

$$\phi_c(t) = \begin{cases} 0 & , \quad 0 \leq t \leq c \\ \left(\frac{2}{3}(t-c)\right)^{3/2} & , \quad c < t \leq 1 \end{cases} \qquad (2)$$

is a solution of (1) on [0, 1]. If the construction of theorem of $\in$- approximate solutions is applied to (1), one find that only polygonal path starting at the point (0, 0) is ϕ_1. This shows that this method cannot, in general, give all solutions of differential equation $\frac{dy}{dt} = f(t, y)$.

1.4.3. Domain. An open connected set in the real (t, y) plane is called a domain.

1.4.4. Theorem. Let $f \in C$ on a domain D in the (t, y) plane, and suppose (t_0, y_0) is any point in D. Then there exists a solution $\phi(t)$ of ordinary differential equation $\frac{dy}{dt} = f(t, y)$ on some t interval containing t_0 in its interior.

Proof. Since D is open, there exist an $r > 0$ such that all points whose distance from (t_0, y_0) is less than r, are contained in D. Let R be any closed rectangle containing (t_0, y_0) and contained in this open circle of radius r. Then applying Cauchy – Peano's existence theorem on R we get the required result.

1.5. Uniqueness of solutions. By the example discussed in remark **3.**, it is clear that something more than the continuity of $f(t, y)$ is required in order to guarantee that a solution passing through a given point be unique.

1.5.1. Lipschitz condition. Suppose f is defined in a domain D in the (t, y) plane. If there exist a constant $K > 0$ such that $|f(t,y_1) - f(t,y_2)| \leq K|y_1 - y_2|$ for every pair of points (t, y_1) and (t, y_2) in D, then the function f is said to satisfy Lipschitz condition w.$r.t$ y in D and constant K is called Lipschitz constant.

The fact that f satisfies Lipschitz condition in the domain D is expressed as $f \in$ Lip in D.

If in addition $f \in C$ in D, $t$hen we write $f \in (C, \text{Lip})$ in D.

Note. If $f \in$ Lip in D then, f is $u$niformly continuous in y for each fixed t.

1.5.2. Example. Show that the function $f(t, y) = y^2$ satisfies Lipschitz condition on the rectangle R defined by $|t| \le a$, $|y| \le b$.

Solution. Here,

$$|f(t, y_1) - f(t, y_2)| = |y_1^2 - y_2^2| = |y_1 + y_2|\ |y_1 - y_2| \qquad (1)$$

We note that maximum value of $|y_1 + y_2|$ in the above defined rectangle is 2b. So by (1), we have

$$|f(t, y_1) - f(t, y_2)| \le 2b|y_1 - y_2| \text{ for all } (t, y_1), (t, y_2) \in R.$$

Thus the given function $f(t, y) = y^{1/2}$ satisfies the Lipschitz condition in the rectangle R with Lipschitz constant 2b.

1.5.3. Theorem. Let $f(t, y)$ be such that $\frac{\partial f}{\partial y}$ exists and is bounded for all $(t, y) \in D$, where D is a domain or closed domain such that the line segment joining any two points of D lies entirely within D . Then f satisfies a Lipschitz condition (w.$r.t.$ y) in d, where the Lipschitx constant is given by

$$K = \underset{(t, y) \in D}{\text{lub}} \left| \frac{\partial f(t, y)}{\partial y} \right|.$$

Proof. Since $\frac{\partial f(t, y)}{\partial y}$ exist and is bounded for all $(t, y) \in D$, there exists a constant $K > 0$, such that

$$K = \underset{(t, y) \in D}{\text{lub}} \left| \frac{\partial f(t, y)}{\partial y} \right|.$$

Moreover, by the mean value theorem of differential calculus, for any pair of points (t, y_1), (t, y_2) in D there exists ξ, $y_1 < \xi < y_2$, such that

$$f(t, y_1) - f(t, y_1) = (y_1 - y_2) \frac{\partial f(t, \xi)}{\partial y} \text{ for all } (t, \xi) \in D$$

Thus, $|f(t, y_1) - f(t, y_2)| = |y_1 - y_2| \left| \frac{\partial f(t, \xi)}{\partial y} \right| \le |y_1 - y_2| \left(\underset{(t, y) \in D}{\text{lub}} \left| \frac{\partial f(t, y)}{\partial y} \right| \right) = K\,|y_1 - y_2|$,

This implies

$$\left|f(t, y_1) - f(t, y_2)\right| \le K\left|y_1 - y_2\right|,$$

for all (t, y_1), (t, y_2) in D. This shows that $f(t, y)$ satisfies a Lipschitz condition in D and K is the Lipschitz constant.

Remark. The sufficient condition of the above theorem is not necessary for $f(t, y)$ to satisfy a Lipschitz condition in d. That is there exists function $f(t, y)$ such that f satisfy a Lipschitz condition in d but such that the hypothesis of theorem is not satisfied.

(a) Consider the function f defined by $f(t, y) = t\left|y\right|$, where d is the rectangle defined by

$$D = \left\{(t, y) : \left|t\right| \le a,\ \left|y\right| \le b\right\}$$

we note that

$$\left|f(t, y_1) - f(t, y_2)\right| = \left|t\left|y_1\right| - t\left|y_2\right|\right| \le \left|t\right|\left|y_1 - y_2\right| \le a\left|y_1 - y_2\right| \text{ for all } (t, y_1) \text{ and } (t, y_2) \text{ in } D$$

Thus $f(t, y)$ satisfies a Lipschaitz condition in D. However, the partial derivative $\frac{\partial f}{\partial y}$ does not exists at any point $(t, 0) \in D$ for which $t \neq 0$.

1.5.4. Exercise.

1. Show that the function $(t, y) = y^{1/2}$ satisfies the Lipschitz condition on any rectangle R: $\left|x\right| \le a$, $b \le y \le c$ (a, b, $c > 0$).
2. Show that the function $f(t, y) = y^{1/2}$ does not satisfy the Lipschitz condition in any domain which includes the line $y = 0$.
3. Using the above theorem, show that following functions satisfy the Lipschitz condition in the rectangle R defined by $\left|t\right| \le a$, $\left|y\right| \le b$

   (i) $f(t, y) = t^2 + y^2$ (ii) $f(t, y) = t - \sin y + y\cos t$.

1.5.5. Theorem. Suppose $f \in (C, \text{Lip})$ in D with Lipschitz constant K. Let ϕ_1 and ϕ_2 be ε_1- and ε_2-approximate solutions of ordinary differential equation

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

in D, of class C'_p on some interval (a, b), satisfying for some t_0, $a < t_0 < b$,

$$\left|\phi_1(t_0) - \phi_2(t_0)\right| \le \delta \tag{2}$$

where δ is a non – negative constant. If $\in\, = \,\in_1 + \in_2$ then for all $t \in (a, b)$

$$\left|\phi_1(t) - \phi_2(t)\right| \le \delta\, e^{K(t-t_0)} + \frac{\in}{K}\left(e^{K(t-t_0)} - 1\right) \tag{3}$$

Proof. We shall consider the case $t_0 \le t < b$ and a corresponding proof holds for $a < t \le t_0$. Since ϕ_1 and ϕ_2 are $\in_1$- and $\in_2$-approximate solutions of ordinary differential **(**1),

$$\left|\phi'_i(s) - f\left(s, \phi_i(s)\right)\right| \le \in_i \quad \text{for} \quad i = 1, 2 \qquad (4)$$

at all points except a finite number of points on the interval $t_0 \le s < b$. Integrating from t_0 to t **(**4), we get

$$\left|\phi_i(t) - \phi_i(t_0) - \int_{t_0}^{t} f\left(s, \phi_i(s)\right) ds\right| \le \in_i (t - t_0), \; i = 1, 2 \qquad (5)$$

Using the fact that $|\alpha - \beta| \le |\alpha| + |\beta|$ and **(**5), we obtain

$$\left|\phi_1(t) - \phi_2(t)\right| - \left|\phi_1(t_0) - \phi_2(t_0)\right| - \left|\int_{t_0}^{t} \left\{f\left(s, \phi_1(s)\right) - f\left(s, \phi_2(s)\right)\right\} ds\right|$$

$$\le \left|\phi_1(t) - \phi_2(t) - \phi_1(t_0) - \phi_2(t_0) - \int_{t_0}^{t} \left\{f\left(s, \phi_1(s)\right) - f\left(s, \phi_2(s)\right)\right\} ds\right|$$

$$= \left|\left\{\phi_1(t) - \phi_1(t_0) - \int_{t_0}^{t} f\left(s, \phi_1(s)\right) ds\right\} - \left\{\phi_2(t) - \phi_2(t_0) - \int_{t_0}^{t} f\left(s, \phi_2(s)\right) ds\right\}\right|$$

$$\le \left|\phi_1(t) - \phi_1(t_0) - \int_{t_0}^{t} f\left(s, \phi_1(s)\right) ds\right| + \left|\phi_2(t) - \phi_2(t_0) - \int_{t_0}^{t} f\left(s, \phi_2(s)\right) ds\right|$$

$$\le \in_1 (t - t_0) + \in_2 (t - t_0) = (\in_1 + \in_2)(t - t_0) = \in (t - t_0) \qquad (6)$$

We define a function $r(t)$ on the interval $t_0 \le t < b$ by the relation $r(t) = \left|\phi_1(t) - \phi_2(t)\right|$. Using this definition in (6) we obtain

$$r(t) \le r(t_0) + \int_{t_0}^{t} \left|f\left(s, \phi_1(s)\right) - f\left(s, \phi_2(s)\right)\right| ds + \in (t - t_0) \qquad (7)$$

Now by hypothesis in the theorem that is, **(**2) we have $r(t_0) \le \delta$. Using this in (7) we obtain,

$$r(t) \le \delta + \int_{t_0}^{t} \left|f\left(s, \phi_1(s)\right) - f\left(s, \phi_2(s)\right)\right| ds + \in (t - t_0) \quad (8)$$

Now *u*sing the fact that $f \in$ Lip in D, we get

$$\left|f\left(s, \phi_1(s)\right) - f\left(s, \phi_2(s)\right)\right| \le K\left|\phi_1(s) - \phi_2(s)\right| = K\, r(s).$$

Using this in (8) we obtain

$$r(t) \leq \delta + K\int_{t_0}^{t} r(s)\,ds + \in (t - t_0) \qquad (9)$$

We define a function $R(t) = \int_{t_0}^{t} r(s)\,ds$, $t_0 \leq t < b$, then $R'(t) = r(t)$. Using this in (9) we obtain

$$r(t) \leq \delta + K R(t) + \in (t - t_0) \qquad (10)$$

and $R'(t) - K R(t) \leq \delta + \in (t - t_0)$ (11)

Multiplying both sides of above inequality (11) by $e^{-K(t-t_0)}$ and then integrating the resulting expression over the interval $[t_0, t]$, we get

$$e^{-K(t-t_0)} R(t) \leq \frac{\delta}{K}\left[1 - e^{-K(t-t_0)}\right] - \frac{\in}{K^2} e^{-K(t-t_0)}\left[1+K(t-t_0)\right] + \frac{\in}{K^2}$$

$$R(t) \leq \frac{\delta}{K}\left[e^{K(t-t_0)} - 1\right] - \frac{\in}{K^2}\left[1+K(t-t_0)\right] + \frac{\in}{K^2} e^{K(t-t_0)}$$

Using this in (10), we obtain

$$r(t) \leq \delta + \delta\left(e^{K(t-t_0)} - 1\right) - \frac{\in}{K}\left[1+K(t-t_0)\right] + \frac{\in}{K} e^{K(t-t_0)} + \in (t-t_0) = \delta e^{K(t-t_0)} + \frac{\in}{K}\left[e^{K(t-t_0)} - 1\right]$$

$$\Rightarrow \left|\phi_1(t) - \phi_2(t)\right| \leq \delta\, e^{K(t-t_0)} + \frac{\in}{K}\left(e^{K(t-t_0)} - 1\right)$$

which is the required result on the interval $[t_0, b]$.

1.5.6. Theorem. Let $f \in (C, \text{Lip})$ in D and $(t_0, y_0) \in D$. If ϕ_1 and ϕ_2 are any two solutions of the initial value problem $\frac{dy}{dt} = f(t, y)$ in D, $y(t_0) = y_0$ on $t \in (a, b)$ such that $\phi_1(t_0) = \phi_2(t_0) = y_0$. Then $\phi_1 = \phi_2$ that is, solution of initial value problem is *u*nique.

Proof. Since ϕ_1 and ϕ_2 are solutions, so $\in_1 = 0$, $\in_2 = 0 \Rightarrow \in = 0 + 0 = 0$.

Also $\delta = 0$ because $\left|\phi_1(t_0) - \phi_1(t_0)\right| = \left|y_0 - y_0\right| = 0$. Hence by above theorem, we must have

$$\left|\phi_1(t) - \phi_2(t)\right| = 0 \text{ for all } t \in (a, b)$$

$$\Rightarrow \phi_1(t) = \phi_2(t) \text{ for all } t \in (a, b) \Rightarrow \phi_1 = \phi_2$$

Note. An another existence proof of our initial value problem depending *u*pon the inequality derived in a previous theorem is as follows.

1.5.7. Theorem. Suppose $f \in$ (C, Lip) on the rectangle R**.** $|t-t_0| \le a,$ $|y-y_0| \le b$ and $a > 0$, $b > 0$. Let M = max. $\{|f(t, y)| : (t, y) \in R\}$ and α = min $\left(a, \dfrac{b}{m}\right)$.Then there exist a *u*nique solution $\phi \in C'$ of the initial value problem

$$\frac{dy}{dt} = f(t, y) \qquad (1)$$

in R, on $|t-t_0| \le \alpha$ for which $y(t_0) = y_0$.

Proof. Let $\{\in_n\}$ be a monotone decreasing sequence of positive real numbers tending to zero, that is, $\in_n \to 0$ as $n \to \infty$. For each $\in_n$ the differential **(**1) has an $\in_n$-approximate solution, ϕ_n. We know that these functions satisfy the relation

$$\phi_n(t) = y_0 + \int_{t_0}^{t} [f(s, \phi_n(s) + \Delta_n(s)] \, ds \qquad (2)$$

where $\Delta_n(s) = \phi_n'(s) - f(s, \phi_n(s))$ (3)

at those points where ϕ_n' exists and $\Delta_n(s) = 0$ otherwise. Now by definition of $\in_n$, we have $\Delta_n(t) \to 0$ as $n \to \infty$, *u*niformly on $|t-t_0| \le \alpha$. Applying inequality of last theorem to the functions ϕ_n and ϕ_m, we obtain

$$|\phi_n(t) - \phi_m(t)| \le \frac{(\in_n + \in_m)(e^{K\alpha} - 1)}{K}$$

where K is the Lipschitz constant and this inequality holds for $|t-t_0| \le \alpha$.

By the above inequality, the sequence $\{\phi_n\}$ is *u*niformly convergent on $|t-t_0| \le \alpha$, and therefore there exists a continuous limit function ϕ on this interval such that $\phi_n(t) \to \phi(t)$ as $n \to \infty$ *u*niformly on $|t-t_0| \le \alpha$.

This fact, plus the *u*niform continuity of f on R, implies that

$$f(t, \phi_n(t)) \to f(t, \phi(t)) \qquad (4)$$

as $n \to \infty$ *u*niformly on the interval $|t-t_0| \le \alpha$. Hence, we get

$$\lim_{n\to\infty} \int_{t_0}^{t} [f(s,\phi_n(s)) + \Delta_n(s)] \, ds = \int_{t_0}^{t} f(s,\phi(s)) \, ds$$

Letting $n \to \infty$ in **(**2) and using **(**4), we obtain

$$\phi(t) = y_0 + \int_{t_0}^{t} f\left(s, \phi(s)\right) ds$$

which proves the existence of a solution $\phi \in C'$ of the initial value problem on $|t - t_0| \leq \alpha$. Also the solution is *u*nique by above theorem because we are given that $f \in (C, \text{Lip})$.

Remark. Clearly, in above theorem $|\phi(t) - \phi_n(t)| \leq \dfrac{\in_n (e^{K(t-t_0)} - 1)}{K}$. This inequality provides a bound for the error in *u*sing $\in_n$-approximate solutions in place of the actual solution. The inequality is valid only when a function f is satisfying the Lipschitz condition.

1.6. Method of successive approximation. Now we shall consider a very *u*seful method for finding a *u*nique exact solution. This method is known as the method of successive approximation.

1.6.1. Picard-Lindelof Theorem.

If $f \in (C, \text{Lip})$ on rectangle R defined by $R = \{(t, y) : |t - t_0| \leq a,\ |y - y_0| \leq b\}$ where $a > 0$, $b > 0$ and (t_0, y_0) is some point in the (t, y) plane. Prove that there are successive approximations ϕ_k, $k = 0, 1, 2, \ldots$, on the interval $|t - t_0| \leq \alpha$ as continuous functions and they converge on this interval to the *u*nique solution ϕ of the initial value problem

$$\frac{dy}{dt} = f(t, y) \text{ in } R,\ y(t_0) = y_0 \qquad (1)$$

with some suitable real number α.

Proof. Since $f \in C$ on the rectangle R so f is bounded. Let $M = \max. \{|f(t, y)| : (t, y) \in R\}$

and $\alpha = \min. \left(a, \dfrac{b}{m}\right)$.

Integrating **(**1) we find an equivalent integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f\left(s, y(s)\right) ds \qquad (2)$$

Thus a solution of problem (1) must satisfy (2) and conversely. Now we define a sequence $\{\phi_k\}$ of successive approximations of the problem (1) by the recurrence formula $\phi_0(t) = y_0$

$$\left.\begin{aligned} \phi_1(t) &= y_0 + \int_{t_0}^{t} f\left(s, \phi_0(s)\right) ds \\ \phi_2(t) &= y_0 + \int_{t_0}^{t} f\left(s, \phi_1(s)\right) ds \end{aligned}\right\} \qquad (3)$$

$$\phi_k(t) = y_0 + \int_{t_0}^{t} f\left(s, \phi_{k-1}(s)\right) ds$$

where $k = 0, 1, 2, \ldots$, on $|t - t_0| \leq \alpha$.

We shall be considering the interval $[t_0, t_0 + \alpha]$ and similar argument holds for the interval $[t_0 - \alpha, t_0]$.

Now, we prove that every

$$\phi_k \in C' \text{ and } |\phi_k(t) - y_0| \leq M(t - t_0) \qquad (4)$$

For $t \in [t_0, t_0 + \alpha]$, we shall prove it by the mathematical induction.

Obviously ϕ_0 satisfy these conditions. Now we assume that ϕ_k does the same and we shall prove the requirements for ϕ_{k+1}. By assumption, $f\left(t, \phi_k(t)\right)$ is defined and continuous on the interval $[t_0, t_0 + \alpha]$. Hence by recurrence formula (3) the function ϕ_{k+1} exists on $[t_0, t_0 + \alpha]$

and $$|\phi_{k+1}(t) - y_0| = \left|\int_{t_0}^{t} f\left(s, \phi_k(s)\right) ds\right| \leq \int_{t_0}^{t} \left|f\left(s, \phi_k(s)\right) ds\right| \leq M(t - t_0)$$

Therefore ϕ_{k+1} satisfy all the necessary requirements. Also $\phi_k \in C'$ clearly. Now we shall prove that the sequence $\{\phi_n\}$ is convergent. For this, we define

$$\Delta_k(t) = |\phi_{k+1}(t) - \phi_k(t)| \text{ for } [t_0, t_0 + \alpha] \qquad (5)$$

Now from (5) and (3), we have

$$\Delta_k(t) = \left|\int_{t_0}^{t} \left[f\left(s, \phi_k(s)\right) - f\left(s, \phi_{k-1}(s)\right)\right] ds\right|$$

$$\leq \int_{t_0}^{t} \left|f\left(s, \phi_k(s)\right) - f\left(s, \phi_{k-1}(s)\right)\right| ds \leq K \int_{t_0}^{t} |\phi_k(s) - \phi_{k-1}(s)| \, ds$$

$$= K \int_{t_0}^{t} \Delta_{k-1}(s) \, ds \qquad (6)$$

where K is the Lip – constant. From (4), we get

$$\Delta_0(t) = |\phi_1(t) - \phi_0(t)| = |\phi_1(t) - y_0| \leq M(t - t_0) \qquad (7)$$

By applying induction on (6), making use of (7), we get

$$\Delta_k(t) \leq \frac{M}{K} \frac{K^{k+1}(t - t_0)^{k+1}}{k+1!}$$

for $k = 1, 2, 3, \ldots$, and $t \in [t_0, t_0 + \alpha]$.

Hence $$\Delta_k(t) \le \frac{M}{K}\ \frac{(\alpha K)^{k+1}}{k+1!} \qquad (8)$$

for all $t \in [t_0, t_0 + \alpha]$.

Since the power series $\sum_{k=0}^{\infty} \frac{M}{K}\ \frac{(\alpha K)^{k+1}}{k+1!}$ is convergent and therefore by Weierstrass – M test, *t*he series $\sum_{k=0}^{\infty} \Delta_k(t)$ is *u*niformly and absolutely convergent on the interval $[t_0, t_0 + \alpha]$. Thus the series $\phi_0(t) + \sum_{k=0}^{\infty} \left(\phi_{k+1}(t) - \phi_k(t)\right)$ is *u*niformly convergent on the interval $[t_0, t_0 + \alpha]$.

Let $$S_n = \phi_0(t) + \sum_{k=0}^{n-1} \left\{\phi_{k+1}(t) - \phi_k(t)\right\} = \phi_n(t)$$

Thus the sequence $\{\phi_n(t)\}$ is absolutely and *u*niformly convergent on the interval $[t_0, t_0 + \alpha]$ to a limit function, say $\phi(t)$, which is continuous on the interval $[t_0, t_0 + \alpha]$.

Finally it will be shown that this limit function $\phi(t)$ is a solution of desired problem. Since $\phi(t)$ is continuous so $f\left(s, \phi(s)\right)$ exists for $s \in [t_0, t_0 + \alpha]$ and

$$\left|\int_{t_0}^{t} \left\{f\left(s,\phi(s)\right) - f\left(s,\phi_k(s)\right)\right\} ds\right| \le \int_{t_0}^{t} \left|f\left(s,\phi(s)\right) - f\left(s,\phi_k(s)\right)\right| ds \le K \int_{t_0}^{t} \left|\phi(s) - \phi_k(s)\right| ds \quad (9)$$

as it is given that function *f* satisfy the Lipschitz condition.

Since $\phi_k \to \phi$ *u*niformly on $[t_0, t_0 + \alpha]$, we have

$$\left|\phi_k(s) - \phi_0(s)\right| \to 0 \qquad (10)$$

as $k \to \infty$ *u*niformly on $[t_0, t_0 + \alpha]$.

Using (9) and (10), we concluded that

$$\int_{t_0}^{t} f\left(s,\phi_k(s)\right) ds \to \int_{t_0}^{t} f\left(s,\phi(s)\right) ds \qquad (11)$$

uniformly on the interval $[t_0, t_0 + \alpha]$ as $k \to \infty$. Using **(**11) in **(**3) and applying the limit $k \to \infty$, we get

$$\phi(t) = y_0 + \int_{t_0}^{t} f\left(s,\phi(s)\right) ds$$

Hence $\phi(t)$ is a solution of integral **(**2) and therefore is a solution of initial value problem (1) on the interval $[t_0, t_0 + \alpha]$. Also *f* satisfies Lipschitz condition and hence $\phi(t)$ is *u*nique also.

Remarks. (1) The inequality (4) geometrically mean that all the functions ϕ_k start at (t_0, y_0) and remain within a triangular region, say T, between the lines $y - y_0 = \pm M(t - t_0)$, t $[t_0, t_0 + \alpha]$.

(2) An *u*pper bound for the error in approximating the solution $\phi(t)$ by the n^{th} – approximation $\phi_n(t)$ is found to be

$$|\phi(t) - \phi_n(t)| = \sum_{k=n}^{\infty} \{\phi_{k+1}(t) - \phi_k(t)\} \le \sum_{k=n}^{\infty} |\phi_{k+1}(t) - \phi_k(t)|$$

$$= \sum_{k=n}^{t} |\Delta_n(t)| \le \sum_{k=n}^{\infty} \frac{M}{K} \frac{(\alpha K)^{k+1}}{k+1!} \qquad \text{[Using (8)]}$$

$$= \frac{M}{K} \sum_{k=n+1}^{\infty} \frac{(\alpha K)^k}{k!} = \frac{M}{K} \frac{(\alpha K)^{n+1}}{n+1!} \sum_{k=0}^{\infty} \frac{(\alpha K)^k}{k!} \le \frac{M}{K} e^{\alpha K} \frac{(\alpha K)^{n+1}}{n+1!}$$

1.6.2. Example. Solve the initial value problem $\frac{dy}{dt} = y$, $y(0) = 1$ by Picard's method.

Solution. The integral equation equivalent to the given initial value problem is

$$y(t) = 1 + \int_0^t y(s)\, ds \qquad (1)$$

The successive approximations given by Picard's method are $\phi_0(t) = 1$

$$\phi_{n+1}(t) = 1 + \int_0^t \phi_n(t)\, dt \qquad (2)$$

for $n = 0, 1, 2, 3, \ldots$, we calculate,

$$\phi_1(t) = 1 + \int_0^t \phi_0(s)\, ds = 1 + \int_0^t 1.\, ds = 1 + t$$

$$\phi_2(t) = 1 + \int_0^t \phi_1(s)\, ds = 1 + \int_0^t (1 + s)\, ds = 1 + t + \frac{t^2}{2!}$$

$$\phi_3(t) = 1 + \int_0^t \phi_2(s)\, ds = 1 + \int_0^t \left(1 + s + \frac{s^2}{2!}\right) ds = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}$$

Continuing like this, we obtain

$$\phi_n(t) = 1 + t + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!}$$

Taking the limit as $n \to \infty$, $\lim_{n\to\infty} \phi_n(t) = e^t$

Therefore, $\phi(t) = e^t$ is the *u*nique solution of given problem.

1.6.3. Exercise.

1. Solve the differential equation $\frac{dy}{dt}$ = ty, $y(0) = 1$ by Picard's method.
2. Solve the initial value problem $\frac{dy}{dt} = t^2y$. $y(0) = 1$ by Picard's method.
3. Solve the initial value problem $\frac{dy}{dt} = y$, $y(1) = 1$ by method of successive approximations.

Solution.

1. $\phi(t) = e^{\frac{t^2}{2}}$.
2. $\phi(t) = e^{\frac{t^3}{3}}$.
3. $\phi(t) = e^{t-1}$.

1.7. Dependence of Solutions on Initial Conditions.

Consider the first order I V P

$$\frac{dy}{dt} = y,$$
$$y(t_0) = y_0. \tag{1}$$

It has the solution (exercise, to obtain it)

$$\varphi(t) = y_0 e^{t-t_0}, \tag{2}$$

which passes through the point (t_0, y_0). The functions φ in (2) can be considered as function, not only $t \in I$, but of the coordinates of point (t_0, y_0), through which the solution curve passes. The solution function φ in (2), without any confusion /ambiguity can be written as

$$\varphi(t, t_0, y_0) = y_0 e^{t-t_0}. \tag{3}$$

Now, we shall investigate the behavior of the solutions as functions of the initial conditions for the general problem.

Let $f(t, y)$ be continuous and satisfy a Lipschitz condition w. r. t. y in a domain D. Let (t_0, y_0) be a fixed point of D. Now, by Picard's existence and uniqueness theorem, the initial value problem

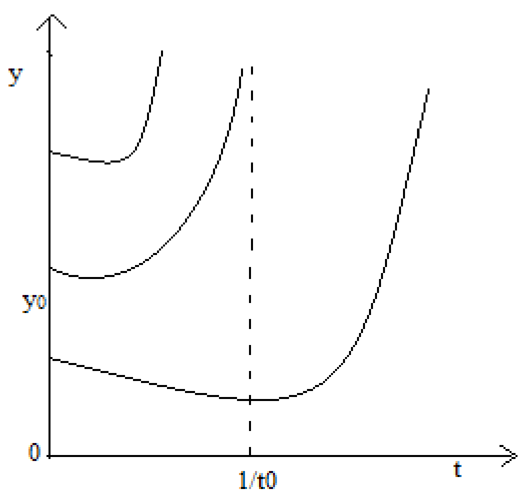
$$\frac{dy}{dt} = f(t, y),$$
$$y(t_0) = y_0, \tag{1}$$

has a unique solution φ defined as some sufficiently small interval [t_0- h_0, t_0+ h_0] around t_0 . Now suppose that the initial y-value is changed from y_0 to Y_0. Our first concern is whether or not the new initial – value problem

$$\frac{dy}{dt} = f(t, y),$$

$$y(t_0) = Y_0, \qquad (2)$$

also has a unique solution on some sufficiently small interval $| t - t_0 | \leq h_1$. If Y_0 is such that $| Y_0 - y_0 |$ is sufficiently small, then we can be certain that the problem (2) does possess a unique solution on some such interval $| t - t_0 | \leq h_1$. In fact, let the rectangle $R: | t - t_0 | \leq a$, $| y - y_0| \leq b$, lie in D and let Y_0 be such that $| Y_0 - y_0 | \leq b/2$. Then, by Picards theorem, this problem has a unique solution y which is defined and contained in R for $| t - t_0 | \leq h_1$, where $h_1 = min(a, b/2M)$ and $M = max | f(t, y) |$ for $(t, y) \in R$. Thus we may assume that there exists $\delta > 0$ and $h > 0$ such that for each Y_0 satisfying $| Y_0 - y_0 | \leq \delta$, problem (2) possesses a unique solution $\varphi(x, Y_0)$ on $|t - t_0| \leq h$ (see Figure below).



Plots of solution for different values of y_0

To illustrate this, consider the I VP

$$\frac{dy}{dt} = ty^3,$$

$$y(0) = y_0,$$

Its solution is

$$\varphi(t) = y_0(1 - {y_0}^2 t^2)^{-1/2},$$

and is only defined for the interval $| t | < | y_0 |_{-1}$. Here, y_0 can be regarded as an arbitrary constant and, as y_0 varies, the solutions fill the entire ty plane. The general solution is shown in the following figure (6.2). Nevertheless, for each particular value y_0, the corresponding unique solution is defined only over an interval whose size depends on y_0.

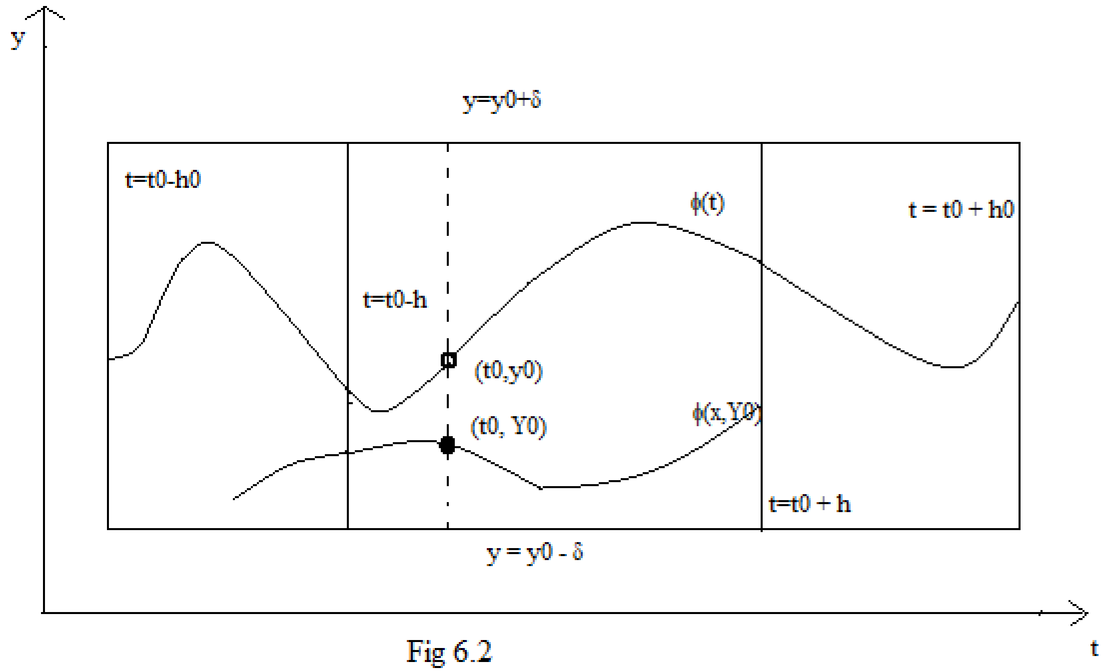


Fig 6.2

We are now in a position to state the basic theorem concerning the dependence of solutions on initial conditions.

1.7.1. Theorem. Let f be continuous and satisfy a Lipschitz condition with respect to y, with Lipschitz constant k, in a domain D of the ty plane; and let (t_0, y_0) be a fixed point of D. Assume there exists $d > 0$ and $h > 0$ such that for each Y_0 satisfying $|\, Y_0 - y_0 \,| \leq d$ the I VP

$$\frac{dy}{dt} = f(t, y),$$

$$y(t_0) = Y_0,$$

possesses a unique solution $f(t, Y_0)$ defined and contained in D on $|\, t - t_0 \,| \leq h$. Let f denote the unique solution of I VP when $Y_0 = y_0$, and $\tilde{\varphi}$ denotes the unique solution of I VP when $Y_0 = \widetilde{y_0}$, where $|\, \widetilde{y_0} - y_0 \,| = d_1 \leq d$. Prove that

$$|\tilde{\varphi}(t) - \varphi(t)| \leq \delta_1 e^{k\square} \quad \text{on } |\, t - t_0 \,| \leq h.$$

Proof. From Picards theorem, we know that

$$\varphi = \lim_{n\to\infty} \varphi_n, \tag{1}$$

where

$$\varphi_n(t) = y_0 + \int_{t_0}^{t} f[t, \varphi_{n-1}(t)]\, dt \qquad (n = 1, 2, 3, \dots), \tag{2}$$

and $\quad \varphi_0(t) = y_0; \quad |t - t_0| \leq h.$

In like manner,

$$\tilde{\varphi} = \lim_{n\to\infty} \tilde{\varphi}_n, \tag{3}$$

where

$$\tilde{\varphi}_n(t) = \tilde{y}_0 + \int_{t_0}^{t} f[t, \tilde{\varphi}_{n-1}(t)]\, dt \qquad (n = 1, 2, 3, \dots), \tag{4}$$

and $\tilde{\varphi}_0(t) = \tilde{y}_0; \quad |t - t_0| \le h.$

We shall show by induction that

$$|\tilde{\varphi}_n(t) - \varphi_n(t)| \le \delta_1 \sum_{j=0}^{n} \frac{K^j (t-t_0)^j}{j} \tag{5}$$

on $[t_0, t_0 + h]$, where K is the Lipschitz constant. We thus assume that on $[t_0, t_0 + h]$,

$$|\tilde{\varphi}_{n-1}(t) - \varphi_{n-1}(t)| \le \delta_1 \sum_{j=0}^{n-1} \frac{K^j (t-t_0)^j}{j} \tag{6}$$

Then

$$|\tilde{\varphi}_n(t) - \varphi_n(t)| = \tilde{\varphi}_n(t) = |\tilde{y}_0 + \int_{t_0}^{t} f[t, \tilde{\varphi}_{n-1}(t)]\, dt - y_0 - \int_{t_0}^{t} f[t, \varphi_{n-1}(t)]\, dt|$$

$$\le |\tilde{y}_0 - y_0| + \int_{t_0}^{t} |f[t, \tilde{\varphi}_{n-1}(t)] - f[t, \varphi_{n-1}(t)]|\, dt.$$

Applying the Lipschitz condition, we have

$$|f[t, \tilde{\varphi}_{n-1}(t)] - f[t, \varphi_{n-1}(t)]| \le K|\tilde{\varphi}_{n-1}(t) - \varphi_{n-1}(t)|$$

and so, since

$$|\tilde{y}_0 - y_0| = \delta_1.$$

Therefore,

$$|\tilde{\varphi}_n(t) - \varphi_n(t)| \le \delta_1 + k \int_{t_0}^{t} |\tilde{\varphi}_{n-1}(t) - \varphi_{n-1}(t)|\, dt. \tag{7}$$

Using the assumption (6), we have

$$|\tilde{\varphi}_n(t) - \varphi_n(t)| \le \delta_1 + k \sum_{j=0}^{n-1} \frac{K^j (t-t_0)^j}{j!}\, dt$$

$$= \delta_1 + k\delta_1 \sum_{j=0}^{n-1} \frac{k^j}{j!} \int_{t_0}^{t} (t-t_0)^j\, dt = \delta_1 \left[1 + \sum_{j=0}^{n-1} \frac{k^{j+1}(t-t_0)^{j+1}}{(j+1)!}\right]. \tag{8}$$

Since

$$\delta_1 \left[1 + \sum_{j=0}^{n-1} \frac{K^{j+1}(t-t_0)^{j+1}}{(j+1)!}\right] = \delta_1 \sum_{j=0}^{n} \frac{K^j (t-t_0)^j}{j!}, \tag{9}$$

We have

$$|\tilde{\varphi}_n(t) - \varphi_n(t)| \le \delta_1 \sum_{j=0}^{n} \frac{K^j (t-t_0)^j}{j!}, \tag{10}$$

which is (6) with $(n-1)$ replaced by n.

Also, on $[t_0, t_0 + h]$, we have

$$|\tilde{\varphi}_1(t) - \varphi_1(t)| = |\tilde{y}_0 + \int_{t_0}^{t} f[t, \tilde{y}_0]\, dt - y_0 - \int_{t_0}^{t} f[t, y_0]\, dt|$$

$$\leq |\tilde{y}_0 - y_0| + \int_{t_0}^{t} |f[t, \tilde{y}_0] - f[t, y_0]|\, dt$$

$$\leq \delta_1 \int_{t_0}^{t} K|\tilde{y}_0 - y_0|\, dt$$

$$= \delta_1 + K\delta_1(t - t_0). \qquad (11)$$

Thus (10.23) holds for $n = 1$. Hence the induction is complete and (5) holds on $[t_0, t_0+h]$. Using similar arguments on $[t_0 - h, t_0]$, we have

$$|\tilde{\varphi}_n(t) - \varphi_n(t)| \leq \delta_1 \sum_{j=0}^{n} \frac{K^j (t-t_0)^j}{j!}$$

$$\leq \delta_1 \sum_{j=0}^{n} \frac{(Kh)^j}{j!}$$

for all t on $|t - t_0| \leq h$, $n = 1, 2, 3, ...$ Letting $n \to \infty$, we have

$$|\tilde{\varphi}(t) - \varphi(t)| \leq \delta_1 \sum_{j=0}^{n} \frac{(Kh)^j}{j!} \qquad (12)$$

But $\sum_{j=0}^{\infty} \frac{(Kh)^j}{j!} = e^{Kh}$, and so we have obtained the desired inequality

$$|\tilde{\varphi}_1(t) - \varphi_1(t)| \leq \delta_1 = e^{Kh} \quad \text{on} \quad |t - t_0| \leq h. \qquad (13)$$

This completes the proof of the theorem.

1.7.2. Corollary. The solution $f(t, Y_0)$ of I V P is a continuous functions of the initial value Y_0 at $Y_0 = y_0$.

Proof :- It follows immediately from the results of the above theorem.

1.8. Check Your Progress.

Solve the following initial value problems by method of successive approximations,

(i) $\frac{dy}{dx} = -t\,y$, $y(0) = 1$.

(ii) $\frac{dy}{dt} = 2y$, $y(0) = 1$

(iii) $\frac{dy}{dt} = t(y - t^2 + 2)$, $y(0) = 1$.

1.9. Summary.

In this chapter, we discussed about various methods to obtain the solution of a given initial value problem. Also, it is observed that whenever the continuous function in initial value problem satisfies Lipschitz condition, the uniqueness of the solution function holds.

Books Suggested:

1. Coddington, E.A., Levinson, N., Theory of ordinary differential equations, Tata McGraw Hill, 2000.
2. Ross, S.L., Differential equations, John Wiley and Sons Inc., New York, 1984.

2

Linear Systems and Second Order Differential Equations

Structure

2.1. Introduction. This chapter contains results related to the properties of solutions of linear systems. The relation between fundamental matrix of a linear homogeneous system and its adjoint system are obtained.

2.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Fundamental set and fundamental matrix of a linear homogeneous system.

(ii) Wronskian of solutions of linear differential equations of order n.

(iii) Relation between the zeros of solution of second order differential equations.

2.1.2. Keywords. Fundamental matrix, Wronskian, Zeros of solutions.

2.2. Basic Definitions.

2.2.1. Norm of a Matrix. Let A is a matrix of complex numbers (a_{ij}) with n rows and n columns then we define the norm of A, denoted by $|A|$, by

$$|A| = \sum_{i,j=1}^{n} |a_{ij}| \quad \text{e.g.} \quad A = \begin{bmatrix} 2+3i & 3-4i \\ 1+i & -i \end{bmatrix}, \ |A| = \sqrt{13} + 5 + \sqrt{2} + \sqrt{1}$$

Note. In case x is an n-dimensional vector that is, $x \in c^n$, represented as a matrix of n rows and one column, then the vector magnitude is defined as

$$|x| = \sum_{i=1}^{n} |x_i| \qquad \text{where } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

It can be easily seen that the norm satisfies the following properties,

(i) $|A+B| \leq |A| + |B|$

(ii) $|AB| \leq |A|.|B|$

(iii) $|Ax| \leq |A|.|x|$

where A and B are $n \times n$ matrices and x is a n-dimensional vector.

2.2.2. Distance Between Two Matrices. The distance between two matrices A and B is defined by $|A-B|$ and distance satisfies the usual properties of metric.

Notation. The zero matrix will be denoted by O and the unit matrix by E. If there is danger of confusion concerning the dimension, these n-by-n matrices will be denoted by O_n and E_n respectively.

Note that $|O_n| = 0$ and $|E_n| = n$.

2.2.3. Complex conjugate of a matrix.

Let $A = [a_{ij}]_{n\times n}$ then $\bar{A} = [\bar{a}_{ij}]_{n\times n.}$

2.2.4. Transpose of a matrix.

Let $A = [a_{ij}]$ then $A^T = [a_{ji}]$

2.2.5. Conjugate transposed of a matrix.

Let $A = [a_{ij}]$ then $A^* = (\bar{A})^T = \overline{(A^T)}$

Note.

(i) $|A^*| = |A^T| = |\overline{A}| = |A|$

(ii) $(AB)^* = B^* A^*$, known as reversal law.

(iii) The determinant of A is denoted by det A. Here norm of A i.e $|A|$ should not be confused with determinant of A.

(iv) The notation A' will be reserved for differentiation, when A is a matrix function. That is why we are using A^T for transpose of matrix A.

(v) If det $A = 0$, then A is said to be singular A non-singular matrix possesses an inverse, A^{-1}, which satisfies $A A^{-1} = A^{-1}A = E$.

2.2.6. Characteristic Roots. Let A be a $n \times n$ complex matrix then det $(\lambda E - A)$, which is a polynomial in λ of degree n, is called the characteristic polynomial of A and its roots are called characteristic roots of A. If these roots are denoted by λ_i, i = 1,…, n, then clearly

$$\det(\lambda E - A) = \prod_{i=1}^{n}(\lambda - \lambda_i)$$

2.2.7. Similar Matrix. Two $n \times n$ complex matrix A and B are said to be similar if there exist a non-singular n-by-n complex matrix P such that $B = PAP^{-1}$.

If A and B are similar, then they have the same characteristic polynomial, for

$$\det(\lambda E - B) = \det(\lambda E - PAP^{-1}) = \left(P(\lambda E - A)P^{-1}\right) = \det P.\det(\lambda E - A).\det P^{-1} = \det(\lambda E - A)$$

Also, if A and B are similar, then

(i) det A = det B

(ii) tr$A$ = TrB, that is, determinant and trace are invariant under similarity transformations.

Now, the following fundamental result concering the canonical form of a matrix is assumed.

2.2.8. Theorem. Every complex n-by-n matrix a is similar to a matrix of the form

$$J = \begin{pmatrix} J_0 & 0 & 0 & \dots & 0 \\ 0 & J_1 & 0 & \dots & 0 \\ 0 & . & . & . & 0 \\ 0 & 0 & 0 & \dots & J_s \end{pmatrix}$$

where J_0 is a diagonal matrix with diagonal entries 1, λ_2, …, λ_q, and

$$J_i = \begin{bmatrix} \lambda_{q+i} & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_{q+i} & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{q+i} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_{q+i} \end{bmatrix}$$

$i = 1, 2, \ldots, s$.

The $\lambda_j, j = 1, 2,\ldots, q + s$, are the characteristic roots of A, which need not be all distinct. If λ_j is a simple root, *t*hen it occurs in J_0, and therefore, if all the roots are distinct, A is similar to the diagonal matrix

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & . & . & . & 0 \\ 0 & 0 & 0 & \ldots & \lambda_s \end{pmatrix}$$

Remark. (i) It follows from above theorem-1 that

$$\det A = \prod \lambda_i \text{ and } \operatorname{tr} A = \sum \lambda_i$$

where the product and sum are taken over all roots, each root counted a number of times equal to multiplicity.

(ii) The $J_i^{'s}$ are of the form $J_i = \lambda_{q+i} E_{r_i} + Z_i$ where J_i has r_i rows and columns, and

$$Z_i = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(iii) If we square the matrix Z_i then it can be found that the matrix Z_i^2 has its diagonal of $1^{'s}$ moved one element to the right from that of Z_i and all other elements zero. From this it follows that Z_i^{r-1} is a matrix which contains all zeros except for a single 1 in the first row and last column. Hence $Z_i^{r_i}$ is the zero matrix and therefore Z_i is nilpotent.

2.2.9. Convergent Sequence of Matrices. Let $\{A_m\}$ is a sequence of matrices then this sequence is said to be convergent if for given any $\in > 0$, there exist a positive integer $N_\in$ such that

$$\left|A_p - A_q\right| < \in \text{ whenever } p, q > N_\in .$$

The sequence $\{A_m\}$ is said to have a limit matrix A, if for given any $\in > 0$, there exist a positive integer $N_\in$ such that

$$\left|A_m - A\right| < \in \text{ whenever } m > N_{\in}.$$

Remark. Clearly {Am} is convergent if and only if each of the component sequences is convergent. Therefore we can say that {Am} is convergent If and only if there exists a limit matrix to which it tends.

The infinite series $\sum_{m=1}^{\infty} A_m$ is said to be convergent if the sequence of partial sums is convergent, and the sum of the series is defined to be the limit matrix of the partial sums.

2.2.10. Exponential of a matrix. *A* particular series which is of great importance for the study of linear equations is the exponential of a matrix *A*, which is defined as

$$e^{A} = E + \sum_{m=1}^{\infty} \frac{A^m}{m!}$$

where A^m represents the m^{th} power of *A*. The series defining e^A is convergent for all *A*, since for any positive integer *p*, *q*,

$$\left|\sum_{m=p+1}^{p+q} \frac{A^m}{m!}\right| \leq \sum_{m=p+1}^{p+q} \frac{\left|A\right|^m}{m!}$$

and the latter represents the Cauchy difference for the series $e^{|A|}$ which is convergent for all finite $\left|A\right|$.

Remark. For matrices, it is not in general true that $e^{A+B} = e^A.e^B$, but this relation is valid if *A* and *B* commute that is, AB = BA. Now since – *A* and *A* commute, so we have

$$e^{A-A} = e^A.e^{-A} \Rightarrow E = e^A.e^{-A} \qquad [\text{Since } e^0 = E]$$

$$\Rightarrow \quad \left(e^A\right)^{-1}.E = \left(e^A\right)^{-1} e^A.e^{-A} \Rightarrow \quad \left(e^A\right)^{-1} = e^{-A}.$$

2.2.11. Caylay – Hamilton Theorem.

Every matrix *A* satisfies its characteristic equation det($\lambda E - A$) = 0. This theorem is sometimes *u*seful for the calculation of e^A. As a simple example, let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ then det } (\lambda E - A) = \lambda^2 = 0$$

is the characteristic equation. So by Caylay – Hamilton theorem, $A^2 = 0$, which implies that $A^m = 0$, $m > 1$. Hence

$$e^A = E + A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

2.2.12. Matrix of Functiions. Let Φ is an *n*-by-*n* matrix of functions defined on a real *t* interval *I* (the functions may be real or complex) then Φ is said to be continuous, differentiable or analytic on *I* if every element of Φ is continuous, differentiable or analytic on *I* respectively.

If Φ is differentiable on I, then Φ' denotes the matrix of derivatives. If Φ and ψ are differentiable then $(\Phi\psi)' = \Phi'\psi + \Phi\psi'$ and also $\Phi'\psi \neq \psi\Phi'$, in general.

2.3. Linear Homogeneous system. Let A be a continuous $n \times n$ matrix of complex functions on a real t interval I. the linear system

$$\frac{dy}{dt} = A(t)\,y,\ t \in I \qquad \text{(LH)}$$

is called a linear homogeneous system of n^{th} order. We have shown in the UNIT-*I* that given any y_0, and $t_0 \in I$, there exists a *u*nique solution ϕ of (LH) on I such that $\phi(t_0) = y_0$.

The zero vector function on I is always a solution of (LH). This will be called the trivial solution of (LH). If any solution of (LH) is zero for any $t_0 \in I$, *t*hen, by *u*niqueness, it must be zero throughout I.

2.3.1. Theorem. The set of all solutions of linear homogeneous system of n^{th} order

$$\frac{dy}{dt} = A(t)\,y,\ t \in I \qquad \text{(LH)}$$

on I form an n-dimensional vector space over complex field.

Proof. Let S be the set of all solutions of (LH). Since $0 \in S$ so S is non-empty.

Let $\phi_1, \phi_2 \in S$ and c_1, c_2 be two complex numbers. Then

$$\frac{d\phi_1}{dt} = A(t)\,\phi_1(t) \text{ and } \frac{d\phi_2}{dt} = A(t)\,\phi_2(t)$$

Now, $\frac{d}{dt}\left[c_1\phi_1 + c_2\phi_2\right] = c_1\frac{d\phi_1}{dt} + c_2\frac{d\phi_2}{dt} = c_1A(t)\phi_1(t) + c_2A(t)\phi_2(t) = A(t)[c_1\phi_1(t) + c_2\phi_2(t)]$

which proves that $c_1\phi_1(t)+c_2\phi_2(t)$ is also a solution of system (LH) and hence $c_1\phi_1+c_2\phi_2 \in S$. Hence S is a vector space.

Now to show that the vector space S is n-dimensional, it is required to establish a set of n linearly independent solution $\phi_1, \phi_2, \ldots, \phi_n$ such that every member of S is a linear combination of $\phi_1, \phi_2, \ldots, \phi_n$.

We know that y – space is n – dimensional. Let ξ_i, $1 \le i \le n$ be n linearly independent points in the n – dimensional y – space. For example, each ξ_i may be taken as a vector with all components zero except the i^{th}, which is 1. Then by existence theorem, if $t_0 \in I$, there exist n solutions $\phi_1, \phi_2, \ldots, \phi_n$ of (LH) such that $\phi_i(t_0) = \xi_i$. Now we shall prove that these solutions satisfy our required conditions.

First we prove that these $\phi_1, \phi_2, \ldots, \phi_n$ are linearly independent. Let, if possible, $\phi_1, \phi_2, \ldots, \phi_n$ are linearly dependent, there must exist n complex numbers $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_n\phi_n(t) = 0 \text{ for } t \in I$$

In particular, *t*aking $t = t_0$, we get

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) + \ldots + c_n\phi_n(t_0) = 0$$

$$\Rightarrow \quad c_1\xi_1 + c_2\xi_2 + \ldots + c_n\xi_n = 0$$

which give that $\xi_1, \xi_2, \ldots, \xi_n$ are linearly dependent, a contradiction to the assumption that $\xi_i^{'s}$ are linearly independent. This shows that $\phi_1, \phi_2, \ldots, \phi_n$ are linearly independent.

Secondly, we shall prove that $\phi_1, \phi_2, \ldots, \phi_n$ generates S. Let $\phi(t)$ be any solution of (LH) on I such that $\phi(t_0) = \xi$. Now ξ belongs to n – dimensional y – space and $\xi_1, \xi_2, \ldots, \xi_n$ is a basis of y – space. So there exist *u*nique scalars (complex numbers) $k_1, k_2, \ldots, k_n$ such that

$$\xi = k_1\xi_1 + k_2\xi_2 + \ldots + k_n\xi_n \qquad (1)$$

Now the function

$$k_1\phi_1(t) + k_2\phi_2(t) + \ldots + k_n\phi_n(t)$$

is a solution of (LH) on I which assumes the value ξ at t_0 as

$$k_1\phi_1(t_0) + k_2\phi_2(t_0) + \ldots + k_n\phi_n(t_0) = k_1\xi_1 + k_2\xi_2 + \ldots + k_n\xi_n = \xi$$

Therefore by the *u*niqueness of solution $k_1\phi_1(t) + k_2\phi_2(t) + \ldots + k_n\phi_n(t)$ must be equal to $\phi(t)$.

Hence every solution $\phi(t)$ of system (LH) is a linear combination of $\phi_1(t), \phi_2(t), \ldots, \phi_n(t)$. Therefore the set $\{\phi_1, \phi_2, \ldots, \phi_n\}$ is a basis of the vector space S, so S is a n – dimensional vector space.

2.3.2. Fundamental set of solutions. If $\phi_1, \phi_2, \ldots, \phi_n$ are a set of n linearly independent solutions of (LH) they are said to form a basis or fundamental set of solutions.

2.3.3. Fundamental matrix. If Φ is a matrix whose n columns are n linearly independent solutions of (LH) on I, then Φ is called a fundamental matrix for (LH). Clearly, fundamental matrix $\Phi(t)$ satisfies the matrix equation

$$\Phi'(t) = A(t)\ \Phi(t),\ t \in I \qquad (1)$$

By the matrix differential equation associated with (LH) on I is meant the problem of finding an n – by – n matrix $\Phi(t)$ whose columns are solutions of (LH) on I. This problem is denoted by

$$X'(t) = A(t)\,X(t),\ t \in I \qquad (2)$$

The matrix $\Phi(t)$ is called a solution of (2) on I and Φ satisfies (1).

From above theorem it is now evident that a complete knowledge of the set of solutions of (LH) can be obtained if we know a fundamental matrix for (LH), which is, of course a solution of (2).

2.3.4. Liouville's Formula. Let $A(t)$ be a $n \times n$ matrix with continuous elements on an interval $I = [a, b]$ and suppose $\Phi(t)$ is a matrix of functions on I satisfying the matrix differential equation

$$\Phi'(t) = A(t)\ \Phi(t),\ t \in I$$

Then det $\Phi(t)$ satisfies the first order equation

$\left(det\ \Phi(t)\right)' = \left(tr\ A(t)\right)\left(det\ \Phi(t)\right)$, on I and for $t_0,\ t \in I$, and

$$\det\ \Phi(t) = \left\{det\ \Phi(t_0)\right\}\ \exp\left[\int_{t_0}^{t} tr\ A(s)\ ds\right].$$

Proof. Let $\Phi(t) = \left[\Phi_{ij}(t)\right]_{n\times n}$ and $A(t) = \left[a_{ij}(t)\right]_{n\times n}$ (1)

Then the given matrix differential equation gives the following scalar differential equations.

$$\phi'_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t)\ \phi_{kj}(t) \tag{2}$$

for i, j = 1, 2,…, n. We know that the derivative of det $\Phi(t)$ is sum of n determinants and given by

$$\left(det\ \Phi(t)\right)' = \begin{vmatrix} \phi'_{11}(t) & \phi'_{12}(t) & \cdots & \phi'_{1n}(t) \\ \phi'_{21}(t) & \phi'_{22}(t) & \cdots & \phi'_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ \phi'_{n1}(t) & \phi'_{n2}(t) & \cdots & \phi'_{nn}(t) \end{vmatrix} + \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) & \cdots & \phi_{1n}(t) \\ \phi'_{21}(t) & \phi'_{22}(t) & \cdots & \phi'_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1}(t) & \phi_{n2}(t) & \cdots & \phi_{nn}(t) \end{vmatrix}$$

$$+\ldots+ \begin{vmatrix} \phi_{11}(t) & \phi_{12}(t) & \cdots & \phi_{1n}(t) \\ \phi_{21}(t) & \phi_{22}(t) & \cdots & \phi_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ \phi'_{n1}(t) & \phi'_{n2}(t) & \cdots & \phi'_{nn}(t) \end{vmatrix} \tag{3}$$

Using (2) in the first determinant on the right, we get

$$\Delta_1 = \begin{vmatrix} \sum_k a_{1k}\phi_{k1} & \sum_k a_{1k}\phi_{k1} & \cdots & \sum_k a_{1k}\phi_{k1} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

and this determinant *u*nchanged if one subtracts from the first row a_{12} times the second row plus a_{13} times the third row *u*p to a_{1n} times the n^{th} row. This gives

$$\Delta_1 = \begin{vmatrix} a_{11}\phi_{11} & a_{11}\phi_{12} & \cdots & a_{11}\phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} = a_{11}\det\Phi(t)$$

Carrying out a similar procedure with the remaining determinants, we get by (3),

$$\left(det\ \Phi(t)\right)' = (a_{11} + a_{22} +\ldots+ a_{nn})\det\ \Phi(t) = (\text{tr}A)\ (\det\phi) \tag{4}$$

This proves the first part of the theorem.

Let $u = \det\ \Phi(t)$, $\alpha(t) = \text{tr}A(t)$. Then equation (4) can be rewritten as

$$\frac{du}{dt} - \alpha(t)u(t) = 0 \text{ or } \frac{du}{u} = \alpha(t)\, dt$$

$\Rightarrow$ $u(t) = C \exp\left[\int_{t_0}^{t} \alpha(s)\, ds\right]$, where C is a constant of integration.

On putting $t = t_0$ both sides, we get $C = u(t_0)$. Hence we get

$$u(t) = u(t_0) \exp\left[\int_{t_0}^{t} \alpha(s)\, ds\right]$$

Consequently, $\det \Phi(t) = \det \Phi(t_0) \exp\left[\int_{t_0}^{t} tr\, \Phi(A)\, ds\right]$ which completes the proof.

2.3.5. Theorem. *A* necessary and sufficient condition that a solution matrix $\Phi(t)$ of

$$X'(t) = A(t)\, X(t) \qquad (1)$$

to be a fundamental matrix of (LH) is that $\det\{\Phi(t)\} \neq 0$ for all $t \in I$.

Proof. Since $\Phi(t)$ is a solution of (1), so

$$\Phi'(t) = A(t)\, \Phi(t)$$

Then we know that

$$\frac{d}{dt}\left[\det \Phi(t)\right] = \mathrm{tr}(A) \det \Phi(t) \qquad (2)$$

and
$$\det \Phi(t) = \det \Phi(t_0) \exp \int_{t_0}^{t} tr\, A(s)\, ds \qquad (3)$$

By relation (3), it is clear that if $\det \Phi(t_0) \neq 0$ then $\det \Phi(t) \neq 0$ for all $t \in I$. Now let $\Phi(t)$ be a fundamental matrix with column vector $\phi_1, \phi_2, \ldots, \phi_n$. Let *u*s suppose that $\phi(t)$ be any non-trivial solution of (LH). Then $\phi(t)$ must be *u*nique linear combination of $\phi_1, \phi_2, \ldots, \phi_n$, so there exist *u*nique constants $c_1, c_2, \ldots, c_n$ (not all zero) such that

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_n\phi_n(t) \qquad (4)$$

Here $\phi(t), \phi_1(t), \ldots, \phi_n(t)$ all are n – by – 1 column matrices of functions therefore relation (4) is a system of n linear equations in the n *u*nknowns $c_1, c_2, \ldots, c_n$ at any $t_0 \in I$ and has a *u*nique solution for any choice of $\phi(t_0)$. Hence $\det \Phi(t_0) \neq 0$ and so we must have by relation (3) that

$$\det \Phi(t) \neq 0 \text{ for any } t \in I.$$

Conversely, let $\Phi(t)$ be a solution matrix of (1) and suppose that $\det \Phi(t) \neq 0$ for all $t \in I$. Then columns vectors of $\Phi(t)$ must be linearly independent at every $t \in I$. Hence by definition of fundamental matrix, $\Phi(t)$ is a fundamental matrix of (LH).

2.3.6. Theorem. If Φ is a fundamental matrix of (LH) that is, $\frac{dy}{dt} = A(t)y,\ t \in I$ and c a complex constant non-singular matrix then Φc is again a fundamental matrix of (LH). Further every fundamental matrix of (LH) is of this type for some non-singular c.

Proof. If Φ is a fundamental matrix then $\Phi'(t) = A(t)\ \Phi(t),\ t \in I$

$$\Rightarrow \Phi'(t)\ c = A(t)\ \Phi(t)c \qquad \Rightarrow \qquad (\Phi c)' = A(t)\ (\Phi\, c)$$

which shows that $\Phi\, c$ is a solution of

$$X'(t) = A(t)\ X(t). \text{ Also, } \det(\Phi\, c) = (\det \Phi)\ (\det c) \neq 0$$

Hence $\Phi\, c$ is a fundamental matrix.

Further, If Φ_1 and Φ_2 are fundamental matrices, then we shall prove that $\Phi_2 = \Phi_1 c$ for some constant non-singular matrix C. To show this, let $\Phi_1^{-1}\Phi_2 = \psi$. Then $\Phi_2 = \Phi_1 \psi$

$$\Rightarrow \qquad \Phi_2' = \Phi_1\psi' + \Phi_I'\,\psi \Rightarrow A\,\Phi_2 = \Phi_1\psi' + A\,\Phi_1\psi \ \left[\mathit{Since}\ \Phi_2^1 = A\Phi_2\ ,\ \Phi_I^I = A\Phi_I\right]$$

$$\Rightarrow \qquad A\,\Phi_2 = \Phi_1\psi' + A\,\Phi_2 \Rightarrow \Phi_1\psi' = 0 \Rightarrow \psi' = 0 \Rightarrow \psi = c \text{ (constant)}.$$

Also c is non-singular since Φ_1 and Φ_2 are non singular. Hence $\Phi_2 = \Phi_1 c$.

2.4. Adjoint System. If Φ be a fundamental matrix for (LH) system, then Φ^{-1} exist and we have

$$\Phi^{-1}\Phi = \Phi\,\Phi^{-1} = E$$

Taking derivative both sides

$$\Rightarrow \qquad \Phi^{-1}\Phi^1 + \left(\Phi^{-1}\right)^1 \Phi = 0 \Rightarrow \left(\Phi^{-1}\right)^1 \Phi = -\,\Phi^{-1}\Phi^1$$

$$\Rightarrow \qquad \left(\Phi^{-1}\right)^1 = -\,\Phi^{-1}\ \Phi^1\,\Phi^{-1} \tag{1}$$

As Φ is a solution of (LH), we must have $\Phi^1 = A\ \Phi$ (2)

putting this value from (2) in equation (1), we get

$$\left(\Phi^{-1}\right)^1 = -\,\Phi^{-1}\,A\ \Phi\ \Phi^{-1} = -\,\Phi^{-1}\,A$$

Taking conjugate transpose both sides

$$\left[\left(\Phi^*\right)^{-1}\right]^1 = -\,A^*\left(\Phi^*\right)^{-1}$$

This equation shows that $\left(\Phi^*\right)^{-1}$ is a fundamental matrix for the system

$$\frac{dy}{dt} = -\,A^*(t)y \tag{3}$$

2.4.1. Adjoint System. The system (3) is called the adjoint to (LH) system and the matrix equation

$$\frac{dY}{dt} = -A^*(t)Y \qquad (4)$$

is called the adjoint to matrix equation $\frac{dY}{dt} = A^*(t)Y$.

2.4.2. Theorem. If Φ is a fundamental matrix of linear homogeneous system

$$\frac{dy}{dt} = A^*(t)y\,,\ t \in I \qquad \text{(LH)}$$

Then ψ is a fundamental matrix for its adjoint

$$\frac{dy}{dt} = -A^*(t)y\,,\ t \in I \qquad \text{(AS)}$$

if and only if $\psi^* \Phi = c$, where c is a constant non-singular matrix.

Proof. As Φ is a fundamental matrix for (LH), so by definition, $\Phi^{*^{-1}}$ is a fundamental matrix for the system (AS). Also we are given that ψ is a fundamental matrix for (AS) system. Then we know that ψ must be of the type

$$\psi = \Phi^{*^{-1}} D \qquad (1)$$

for some constant non-singular matrix D. Pre-multiplying (1) both sides by Φ^*,

$$\Phi^* \psi = D$$

Taking conjugate transpose both sides

$$\psi^* \Phi = D^* = c \text{ (say)}.$$

Conversely. Let *u*s suppose that given condition is satisfied that is, $\psi^* \Phi = c$

Post multiplying both sides by Φ^{-1},

$$\Rightarrow \qquad \psi^* = c\ \Phi^{-1}$$

Taking conjugate transpose both sides

$$\psi = \Phi^{*^{-1}}\ c^*$$

But $\Phi^{*^{-1}}$ is a fundamental matrix for (AS) system by definition. Hence by above theorem, it follows that ψ is also a fundamental matrix for (AS).

Remark.

1. If matrix A is such that $A = -A^*$ that is, A is skew – hermitian. Then clearly, both the systems that is, (LH) system and (AS) system becomes same. Since $\Phi^{*^{-1}}$ is a fundamental matrix for (AS), then it must be a fundamental matrix of (LH) also. Hence Φ, which is already a fundamental matrix for (LH), must be of the type $\Phi = \Phi^{*^{-1}}\ C, c \rightarrow$ non – singular constant matrix

$$\Rightarrow \quad \Phi^* \Phi = c.$$

2. We know that if Φ is a fundamental matrix for (LH) and c is a non – singular constant matrix then Φc is also a fundamental matrix for (LH). Here it should be noted that $c\Phi$ is not a fundamental matrix of (LH), in general.

3. Two different homogeneous systems can not have the same fundamental matrix. Since if Φ is a fundamental matrix for (LH), then

$$\Phi^1 = A(t)\ \Phi \quad \Rightarrow A(t) = \Phi^1(t)\ \Phi^{-1}(t)$$

Hence Φ determines a *u*niquely, although the converse is not true.

2.5. Non-Homogeneous Linear System. Suppose $A(t)$ is an n-by-n matrix of continuous functions on a real t interval, and $b(t)$ is a continuous vector on I which is not identically zero there. The system

$$\frac{dy}{dt} = A(t)y + b(t),\ t \in I \qquad \text{(NH)}$$

is called a non-homogeneous linear system of n^{th} order. If the elements of A and b are continuous, *t*hen there exists a *u*nique solution ϕ of (NH) for which $\phi(t_0) = y_0$ where $t_0 \in I$ and $|y_0| < \infty$.

For *u*niqueness, let ϕ_1 and ϕ_2 be two solution of (NH) such that $\phi_1(t_0) = y_0$, $\phi_2(t_0) = y_0$, then their difference $\phi = \phi_1 - \phi_2$ would be a solution of (LH) on I and would satisfy $\phi(t_0) = 0$. But by the *u*niqueness theorem for (LH), $\phi(t)$ must be identically zero function on I and thus $\phi_1(t) = \phi_2(t)$ for all $t \in I$.

Note. If a fundamental matrix Φ for (LH) is known, *t*hen there is a simple method for calculating a solution of (NH).

2.5.1. Theorem. If Φ is a fundamental matrix for (LH) system

$$\frac{dy}{dt} = A(t)y,\ t \in I \qquad \text{(LH)}$$

where $A(t)$ is a $n \times n$ matrix, *t*hen the function defined by

$$\phi(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)\, b(s)\, ds\ ,\ t \in I. \qquad (1)$$

is a solution of non-homogeneous linear system

$$\frac{dy}{dt} = A(t)y + b(t),\ t \in I \qquad \text{(NH)}$$

satisfying the initial condition $\phi(t_0) = 0$, $t_0 \in I$.

Proof. We know that linear combination of solutions of (LH) is also a solution of (LH). Therefore for any constant vector c, *t*he function Φc is a solution of (LH), here c is a $n \times 1$ matrix. We shall consider c as a function of t on I and determine what c must be (if it exists) in order that the function $\phi = \Phi c$ be a solution of the non-homogeneous system (NH).

Suppose $\phi = \Phi c$ is a solution of (NH). Then, since $\Phi' = A\Phi$

$$\phi'(t) = \Phi'(t)c(t) + \Phi(t)c'(t) = A(t)\Phi(t)\,c(t) + \Phi(t)c'(t) = A(t)\phi(t) + \Phi(t)c'(t) \quad (2)$$

Also as ϕ is a solution of (NH), so

$$\phi'(t) = A(t)\,\phi(t) + b(t) \qquad (3)$$

Comparing (2) and (3), we have

$$\Phi(t)\;c'(t) = b(t) \qquad \Rightarrow\; c'(t) = \Phi^{-1}(t)\,b(t)$$

Integrating both sides, we get

$$c(t) = \int_{t_0}^{t} \Phi^{-1}(s)\,b(s)\,ds\,,\ t_0 \in I$$

Here, we assume that $C(t_0) = 0$. Hence, we have $\phi = \Phi c = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\,b(s)\,ds$, is a solution of (NH) with the condition $\phi(t_0) = 0$.

Remarks 1. The formula (1) to find the solution of (NH) is called the variation – of – constants formula for (NH).

(2) If Φ is a fundamental matrix for (LH), *t*hen it can be easily seen that the solution $\phi(t)$ of (NH) which satisfies the initial condition $\phi(t_0) = y_0$ is given by

$$\phi(t) = \phi_h(t) + \phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\,b(s)\,ds\,,\ t \in I$$

where $\phi_h(t)$ is that solution of (LH) on I which satisfies $\phi_h(t_0) = y_0$.

(3) If ψ is a fundamental matrix for the adjoint system

$$\frac{dy}{dt} = -\,A^*(t)y \qquad \text{(AS)}$$

then the solution ϕ of (NH) may be written as

$$\phi(t) = \psi^{*^{-1}}(t)\int_{t_0}^{t} \psi^*(s)\,b(s)\,ds\,,\ t \in I.$$

Here it should be noted that if ψ is a fundamental matrix for (AS) then $\psi^{*^{-1}}(t)$ is for (LH).

2.6. Linear systems with constant coefficients. Let A be an $n \times n$ constant matrix and consider the corresponding homogeneous system

$$\frac{dy}{dt} = Ay \qquad (1)$$

If $n = 1$, *t*hen it is trivial that (1) has a solution e^{tA} and the solution satisfying the initial condition $\phi(t_0) = y_0$ $\left(|t_0| < \infty\,,\ |y_0| < \infty\right)$ is given by

$$\phi(t) = y_0\ e^{(t-t_0)A}$$

Here it should be noted that y and y_0 are vectors of arbitrary finite dimension n, and A is an $n \times n$ matrix.

2.6.1. Theorem. *A* fundamental matrix Φ for the system

$$\frac{dy}{dt} = Ay \qquad (1)$$

is given by $\Phi(t) = e^{tA},\ |t| < \infty$ (2)

and the solution ϕ of (1) satisfying the initial condition $\phi(t_0) = y_0$ $\left(|t_0| < \infty\ ,\ |y_0| < \infty\right)$ is given by

$$\phi(t) = e^{(t-t_0)A}\ y_0 \qquad \left(|t| < \infty\right) \qquad (3)$$

Proof. We have

$$e^{(t+\Delta t)A} = e^{tA}\ e^{\Delta tA} \Rightarrow \frac{e^{(t+\Delta t)A} - e^{tA}}{\Delta t} = \frac{e^{tA}\ e^{\Delta tA} - e^{tA}}{\Delta t}$$

$$\Rightarrow \qquad \frac{e^{(t+\Delta t)A} - e^{tA}}{\Delta t} = \left(\frac{e^{\Delta tA} - E}{\Delta t}\right)e^{tA}$$

Taking limits $\Delta t \to 0$ both sides

$$\lim_{\Delta t\to 0} \frac{e^{(t+\Delta t)A} - e^{tA}}{\Delta t} = \left(\lim_{\Delta t\to 0}\frac{e^{\Delta tA} - E}{\Delta t}\right)e^{tA} \Rightarrow \frac{d}{dt}\left(e^{tA}\right) = A\ e^{tA}$$

which shows that $\Phi(t) = e^{tA}$ is a solution of system (1). Now to prove that $\Phi(t)$ is a fundamental matrix it remains to prove that $\det\Phi(t) \neq 0$ " we know that if Φ is solution, that is, Φ satisfies $\Phi'(t) = a(t)\Phi(t)$ on I then $\det\Phi(t) = \det\Phi(t_0)\exp\int_{t_0}^{t} \text{tr } A(s)\, ds$, $t, t_0 \in I$ ".

Using this reslt for the given system and for $t_0 = 0$ to otain

$$\det\ \Phi(t) = \det\ \Phi(0)\ \exp\int_{t_0}^{t} \text{tr } A(s)\, ds$$

Since $\Phi(0) = E$ so det $\Phi(t_0) = \det E = 1$

Therefore,

$$\det\Phi(t) = \exp\int_{t_0}^{t} \text{tr } A.ds = \exp \text{tr } A\int_{t_0}^{t} ds = \exp\ t\ \text{trA} = e^{t\ \text{trA}}.$$

Thus Φ is a fundamental matrix.

Also it is obvious that solution passing through (t_0, y_0) that is, satisfying the initial condition $\phi(t_0) = y_0$ is given by

$$\phi(t) = e^{(t-t_0)A}\ y_0.$$

2.7. Linear Differential Equations of Order *n*.

Suppose $a_0, a_1, \ldots, a_n$ are $(n+1)$ continuous (complex) functions defined on a real t interval I, and let L_n denote the formal differential operator

$$L_n = a_0\frac{d^n}{dt^n} + a_1\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_n.$$

If g is any function possessing n derivatives on I,

$$L_n g = a_0\frac{d^n g}{dt^n} + a_1\frac{d^{n-1}g}{dt^{n-1}} + \ldots + a_n g.$$

The differential equation $L_n y = 0$, $t \in I$ is defined to be the differential equation, $a_0(t) \neq 0$

$$y^{(n)} + \frac{a_1(t)}{a_0(t)}y^{(n-1)} + \ldots + \frac{a_n(t)}{a_0(t)}y = 0, \qquad t \in I \tag{1}$$

and is called a linear homogeneous differential equation of order n. The system associated with this equation (1) is then the vector equation

$$\frac{d\hat{y}}{dt} = A(t)\,\hat{y} \tag{2}$$

$$\text{where } A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \frac{-a_n}{a_0} & \frac{-a_{n-1}}{a_0} & \cdots & \cdots & \cdots & \frac{-a_1}{a_0} \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n-1)} \end{bmatrix} \tag{3}$$

Since (2) is a linear system with a continuous coefficient matrix $A(t)$ on I so there exist a unique vector solution $\hat{\phi}$ of (2) on I satisfying

$$\hat{\phi}(t_0) = \hat{y}_0$$

where $t_0 \in I$, $\left|\hat{y}_0\right| < \infty$. Thus ϕ_1, the first component of $\hat{\phi}$, satisfies

$$\phi_1(t_0) = y_0^1, \phi_1^{(1)}(t_0) = y_0^2, \ldots, \phi_1^{(n-1)}(t_0) = y_0^n.$$

2.7.1. Wronskian. If $\phi_1, \phi_2, \ldots, \phi_n$ are n solutions of $L_n y = 0$, then the matrix

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \phi_n \\ \phi'_1 & \phi'_2 & \phi'_n \\ \vdots & \vdots & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \phi_n^{(n-1)} \end{bmatrix}$$

is a solution matrix of (2). The determinant of this matrix is called the Wronskian of $L_n\, y = 0$ with respect to $\phi_1, \phi_2, \ldots, \phi_n$ and is denoted by $W(\phi_1, \phi_2, \ldots, \phi_n)$. It is a function of t on I for fixed $\phi_1, \phi_2, \ldots, \phi_n$ and its value at t is denoted by $W(\phi_1, \phi_2, \ldots, \phi_n)(t)$. We know that for the system (2), we have

$$\det \Phi(t) = \det \Phi(t_0) \exp \int_{t_0}^{t} \operatorname{tr} A(s)\, ds,\ t \in I \qquad (4)$$

By (3), we note that $\operatorname{tr} A(t) = \dfrac{-a_1}{a_0}$

Using this value of tr A and definition of Wronskian, we get

$$W(\phi_1, \phi_2, \ldots, \phi_n(t)) = W(\phi_1, \phi_2, \ldots, \phi_n)(t_0) \exp \int_{t_0}^{t} -\frac{-a_1(s)}{a_0(s)}\, ds,\ t \in I$$

Remark. From above equation, it follows that if $W(\phi_1, \phi_2, \ldots, \phi_n)(t_0) = 0$ for some $t_0 \in I$, then $W(\phi_1, \phi_2, \ldots, \phi_n)(t) = 0$ for all $t \in I$. Also if $W(\phi_1, \phi_2, \ldots, \phi_n)(t_0) \neq 0$ for some $t_0 \in I$ then $W(\phi_1, \phi_2, \ldots, \phi_n)(t) \neq 0$, for all $t \in I$, because $\exp \int_{t_0}^{t} -\frac{-a_1(s)}{a_0(s)}\, ds$ can never be zero. So, we can say that Wronskian of n solutions is either identically zero or no where zero on the interval I.

2.7.2. Theorem. *A* necessary and sufficient condition that n solutions $\phi_1, \phi_2, \ldots, \phi_n$ of differential equation $L_n\, y = 0$, that is, $a_0(t) \dfrac{d^n y}{dt^n} + a_1(t) \dfrac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n(t)\, y = 0$, $a_0(t) \neq 0$ to be linearly dependent on interval I is that $W(\phi_1, \phi_2, \ldots, \phi_n)(t) = 0$ for all $t \in I$.

Proof. Let $\phi_1, \phi_2, \ldots, \phi_n$ be linearly dependent on I. Then there exist constants $c_1, c_2, \ldots, c_n$ (not all zero) such that

$$c_1\phi_1(t) + c_2\phi_2(t) + \ldots + c_n\phi_n(t) = 0 \text{ for all } t \in I$$

Also then,

$$c_1\phi_1^{(1)}(t) + c_2\phi_2^{(1)}(t) + \ldots + c_n\phi_n^{(1)}(t) = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$c_1\phi_1^{(n-1)}(t) + c_2\phi_2^{(n-1)}(t) + \ldots + c_n\phi_n^{(n-1)}(t) = 0 \text{ for all } t \in I.$$

For any fixed $t_0 \in I$, these equations are linear homogeneous equations which are satisfied by

$c_1, c_2, \ldots, c_n$ (not all zero). So, *t*he determinant of the coefficients of $c_1, c_2, \ldots, c_n$ that is, $W(\phi_1, \phi_2, \ldots, \phi_n)(t_0) = 0$. Then we must have $W(\phi_1, \phi_2, \ldots, \phi_n)(t) = 0$ for all $t \in I$.

Conversely, let *u*s suppose that $W(\phi_1, \phi_2, \ldots, \phi_n)(t) = 0$ for all $t \in I$. For any $t_0 \in I$, *t*he system linear equations

$$c_1\phi_1(t_0) + c_2\phi_2(t_0) + \ldots + c_n\phi_n(t_0) = 0$$

$$c_1\phi_1^{(1)}(t_0) + c_2\phi_2^{(1)}(t_0) + ... + c_n\phi_n^{(1)}(t_0) = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$c_1\phi_1^{(n-1)}(t_0) + c_2\phi_2^{(n-1)}(t_0) + ... + c_n\phi_n^{(n-1)}(t_0) = 0.$$

must have a non-trivial solution. Let $k_1, k_2, \ldots, k_n$ be such a solution. Then we consider the function

$$\phi = k_1\phi_1(x) + k_2\phi_2(x) + \ldots + k_n\phi_n(x) \qquad (1)$$

Since $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ are solutions of the given homogeneous linear differential equation, therefore $\phi(x)$ given by (1) is also a solution. Due to the system of linear eqautions, we have

$$\phi(t_0) = 0,\ \phi'_1(t_0) = 0, \ldots, \phi^{(n-1)}(t_0) = 0.$$

Therefore, by *u*niqueness theorem.

$$\phi(t) = 0 \text{ for all } t \in I$$

$$\Rightarrow \quad k_1\phi_1(t) + k_2\phi_2(t) + \ldots + k_n\phi_n(t) = 0 \text{ for all } t \in I$$

$\Rightarrow$ $\phi_1(t), \phi_2(t), \ldots, \phi_n(t)$ are linearly dependent on the interval I as $k_1, k_2, \ldots, k_n$ are not all zero. Hence the theorem.

2.7.3. Corollary. *A* necessary and sufficient condition that n solutions $\phi_1, \phi_2, \ldots, \phi_n$ of $L_n y = 0$ on an interval I be linearly independent there is that

$$W(\phi_1, \phi_2, \ldots, \phi_n)(t) \neq 0, \text{ for all } t \in I$$

Proof. Let $\phi_1, \phi_2, \ldots, \phi_n$ are linearly independent. Let, if possible, $W(\phi_1, \phi_2, \ldots, \phi_n)(t) = 0$ for all $t \in I$. Then, by above theorem, $\phi_1, \phi_2, \ldots, \phi_n$ would become linearly dependent, a contradiction. Hence we must have $W(\phi_1, \phi_2, \ldots, \phi_n)(t) \neq 0$, for all $t \in I$.

Conversely, let *u*s suppose that $W(\phi_1, \phi_2, \ldots, \phi_n)(t) \neq 0$, for all $t \in I$. We have to prove that $\phi_1, \phi_2, \ldots, \phi_n$ are linearly independent. Let, if possible, $\phi_1, \phi_2, \ldots, \phi_n$ are linearly dependent. Then by above theorem, wronskian must be zero on interval I, which is not so. So

$\phi_1, \phi_2, \ldots, \phi_n$ must be linearly independent.

2.7.4. Theorem. Every solution of $L_n y = 0$ is a linear combination with complex coefficients of any n linearly independent solutions.

Proof. We know that the equation $L_n y = 0$ is of the type

$$\frac{d\hat{y}}{dt} = A(t)\,\hat{y} \qquad (1)$$

where $$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \frac{-a_n}{a_0} & \frac{-a_{n-1}}{a_0} & \cdots & \cdots & \cdots & \frac{-a_1}{a_0} \end{bmatrix}, \ \hat{y} = \begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

Now, we have proved the set of solutions of linear homogeneous system of n^{th} order, that is, (1) form an *n*-dimensional vector space over complex field. So every solution vector of (1) is a linear combination of *n* linearly independent vector solution and, therefore, every solution of $L_n\, y = 0$ is a linear combination of *n* linearly independent solutions of $L_n\, y = 0$.

This proves the theorem.

2.7.5. Fundamental Set. *A* set of *n* linearly independent solutions of $L_n\, y = 0$ is called a basis or fundamental set of $L_n\, y = 0$.

2.7.6.Theorem. Suppose $\phi_1, \phi_2, \ldots, \phi_n$ are *n* functions which possess continuous n^{th} order derivatives of a real *t* interval *I*, and $W(\phi_1, \phi_2, \ldots, \phi_n)(t) \neq 0$ for all $t \in I$. Then there exist a unique homogeneous differential equation of order *n* (with coefficient of $y^{(n)}$ one) for which these functions form a fundamental set, namely

$$(-1)^n \frac{W(y, \phi_1, \phi_2, \ldots, \phi_n)}{W(\phi_1, \phi_2, \ldots, \phi_n)} = 0$$

Proof. Consider the differential equation $W(\phi_1, \phi_2, \ldots, \phi_n) = 0$, that is,

$$\begin{vmatrix} y & \phi_1 & \phi_2 & \cdots & \phi_n \\ y' & \phi'_1 & \phi'_2 & \cdots & \phi'_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y^{(n)} & \phi_1^{(n)} & \phi_2^{(n)} & \cdots & \phi_n^{(n)} \end{vmatrix} = 0 \qquad (1)$$

$t \in I$. If we expand this determinant by first column, we see that the coefficient of $\frac{d^n y}{dt^n}$ is

$$(-1)^n \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi'_1 & \phi'_2 & \cdots & \phi'_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} = (-1)^n\, W(\phi_1, \phi_2, \ldots, \phi_n)$$

which is not zero on the interval I by the given hypothesis of the theorem. Therefore equation (1) is a n^{th} order linear homogeneous differential equation. If we divide differential equation (1) by $(-1)^n W(\phi_1, \phi_2, \ldots, \phi_n)$ then coefficient of $y^{(n)}$ would become *u*nity, and differential equation would become

$$(-1)^n \frac{W(y, \phi_1, \phi_2, \ldots, \phi_n)}{W(\phi_1, \phi_2, \ldots, \phi_n)} = 0 \qquad (2)$$

which is the required type of differential equation.

Further it is clear from equation (1) that $\phi_1, \phi_2, \ldots, \phi_n$ are solutions of differential equation (1) as two columns of (1) become identical. Moreover, $W(\phi_1, \phi_2, \ldots, \phi_n)(t) \neq 0$ for all t in I, so $\phi_1, \phi_2, \ldots, \phi_n$ are linearly independent. Hence solution $\phi_1, \phi_2, \ldots, \phi_n$ from a fundamental set of differential equation (*) on interval I.

The uniqueness of (*) follows from the fact that the corresponding vectors $\hat{\phi}_i$ (i = 1, 2, …, n) with components $\phi_i, \phi_i', \ldots, \phi_i^{(n-1)}$ determine the coefficient matrix $A(t)$ of the associated system

$$\frac{d\hat{y}}{dt} = A(t)\,\hat{y} \qquad (3)$$

where

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \frac{-a_n}{a_0} & \frac{-a_{n-1}}{a_0} & \cdots & \cdots & \cdots & \frac{-a_1}{a_0} \end{bmatrix} \qquad (4)$$

uniquely. Since there is a one – to – one correspondence between linear equation of order n and linear system of the type (3) and (4), the proof is complete.

2.8. Adjoint Equations. Connected with the formal operator

$$L_n = a_0(t)\frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_n(t),$$

$t \in I$, there is another linear operator of order n denoted by L_n^+, called the adjoint of L_n, as

$$L_n^+ = (-1)^n \left(\frac{d^n}{dt^n}\right)\left\{\overline{a_0(t)}\right\} + (-1)^{n-1}\left(\frac{d^{n-1}}{dt^{n-1}}\right)\left\{\overline{a_1(t)}\right\} + \ldots + \overline{a_n(t)}.$$

If $g(t)$ is any function on I which is such that $\overline{a_k(t)}\,g(t)$ ($k = 0, 1, 2, \ldots, n$) has $n - k$ derivatives on I, then

$$L_n^+ g = (-1)^n \frac{d^n}{dt^n}\left(\overline{a_0(t)}\,g(t)\right) + (-1)^{n-1}\frac{d^{n-1}}{dt^{n-1}}\left(\overline{a_1(t)}\,g(t)\right) + \ldots + \overline{a_n(t)}\,g(t)$$

The equation $L_n^+ y = 0$, $t \in I$ is called the adjoint equation to the differential equation $L_n\, y = 0$ on interval I. For example, let us find adjoint equation for second order differential equation

$$L_2 y = a_0(t)\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)\, y = 0 \qquad (1)$$

where $a_k(t) \in C^{n-k}$ and a_0, a_1, a_2 are real valued functions $k = 0, 1, 2$, that is, $a_0(t) \in c^2$, $a_1(t) \in c^1$ and $a_2(t) \in c$ and let $a_0(t) \neq 0$ on the considered interval. Then, adjoint equation of (1) is given by

$$(-1)^2 \frac{d^2}{dt^2}\left[a_0(t)y\right] + (-1)' \frac{d}{dt}\left[a_1(t)y\right] + a_2(t)y = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left[a_0(t)\frac{dy}{dt} + a'_0(t)y\right] - a_1(t)\frac{dy}{dt} - a'_1(t)y + a_2(t)y = 0$$

$$\Rightarrow \quad a_0(t)\frac{d^2y}{dt^2} + a'_0(t)\frac{dy}{dt} + a'_0(t)\frac{dy}{dt} + a''_0(t)y - a_1(t)\frac{dy}{dt} - a'_1(t)y + a_2(t)y = 0$$

$$\Rightarrow \quad a_0(t)\frac{d^2y}{dt^2} + \left[2a'_0(t) - a_1(t)\right]\frac{dy}{dt} + \left[a''_0(t)y - a'_1(t) + a_2(t)\right]y = 0 \qquad (2)$$

which is the required adjoint equation. For example,

$$L_2\, y = t^2 \frac{d^2y}{dt^2} + 7t \frac{dy}{dt} + 8y = 0.$$

Here $a_0(t) = t^2$, $a_1(t) = 7t$, $a_2(t) = 8$ putting these values in equation (2), we get

$$t^2 \frac{d^2y}{dt^2} - 3t \frac{dy}{dt} - 3y = 0.$$

Special case. Consider the special case of L_n where $a_0(t) = 1$, then

$$L_n y = \frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1(t)y = 0 \qquad (1)$$

for all $t \in I$. We know that the associated linear system with (1) is

$$\frac{d\hat{y}}{dt} = A(t)\ \hat{y} \qquad (2)$$

where

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix} \qquad (3)$$

The adjoint system for (2) is

$$\frac{d\hat{y}}{dt} = -\ A^*(t)\ \hat{y} \tag{4}$$

where

$$A^*(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \bar{a}_n \\ -1 & 0 & \cdots & 0 & \bar{a}_{n-1} \\ 0 & -1 & \cdots & 0 & \bar{a}_{n-2} \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & -1 & \bar{a}_1 \end{bmatrix} \tag{5}$$

By (4) and (5), we have

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \bar{a}_n \\ -1 & 0 & \cdots & 0 & \bar{a}_{n-1} \\ 0 & -1 & \cdots & 0 & \bar{a}_{n-2} \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & -1 & \bar{a}_1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \text{ where } \hat{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Equating these two we get

$$y'_1 = \bar{a}_n\, y_n,\ y'_k = -y_{k-1} + \bar{a}_{n-k+1}\ y_n,\ (k = 2, \ldots, n) \tag{6}$$

Thus if $\phi_1, \phi_2, \ldots, \phi_n$ is a solution of (6) for which $\phi_k^{(k)}$ and $\left(\bar{a}_{n-k+1}\right)^{(k-1)}$ exist, *t*hen we have

$$\left.\begin{aligned} \phi'_1 &= \bar{a}_n\ \ \phi_n \\ \phi'_2 &= -\ \phi_1 + \bar{a}_{n-1}\ \ \phi_n \\ \phi'_3 &= -\ \phi_2 + \bar{a}_{n-2}\ \ \phi_n \\ &\vdots \\ \phi'_n &= -\ \phi_{n-1} + \bar{a}_0\ \ \phi_n \end{aligned}\right\} \tag{7}$$

Now differentiating kth relation of (7) (k–1) times and solving for $\phi_n^{(n)}$, we get

$$\phi_n^{(n)} - \left(\bar{a}_1\phi_1\right)^{(n-1)} + \ldots . + (-1)^n \left(\bar{a}_n\phi_n\right) = 0 \tag{8}$$

By equation (8), we can say that ϕ_n satisfies the equation $L_n^+ y = 0,$ that is,

$$(-1)^n\, y^{(n)} + (-1)^{n-1}\left(\bar{a}_1\ y\right)^{n-1} + \ldots + \bar{a}_n\, y = 0$$

which is just the adjoint equation of (1).

2.8.1. Lagrange's Identity. In $L_n = a_0(t)\dfrac{d^n}{dt^n} + a_1(t)\dfrac{d^{n-1}}{dt^{n-1}} + \ldots + a_n(t)$, suppose $a_k \in C^{n-k}$ on I for $k = 0, 1, 2, \ldots, n$. If u and v are any two (complex) functions on I possessing n derivatives there, then

$$\bar{v}\, L_n u - u\, \overline{L_n^+ v} = \frac{d}{dt}\,[P(u, v)]$$

where $P(u, v)$ is a form in $(u, u', \ldots, u^{(n-1)})$ and $(v, v', \ldots, v^{(n-1)})$ given by

$$P(u, v) = \sum_{m=1}^{n}\left[\sum_{j=1}^{m} (-1)^{j-1}\, u^{(m-j)} \left(a_{n-m}\bar{v}\right)^{(j-1)}\right]$$

Proof. If $U(t)$ and $V(t)$ be any two functions, then

$$\frac{d}{dt}\left[U^{(m-1)}V - U^{(m-2)}V^{(1)} + \ldots + (-1)^{(m-2)}U^{(1)}V^{(m-2)} + (-1)^{(m-1)}UV^{(m-1)}\right]$$

$$= U^{(m)}V + U^{(m-1)}V^{(1)} - U^{(m-1)}V^{(1)} - U^{(m-2)}V^{(2)} + \ldots + (-1)^{m-2}U^{(2)}V^{(m-2)} + (-1)^{m-2}U^{(1)}V^{(m-1)}$$
$$+ (-1)^{m-1}U^{(1)}V^{(m-1)} + (-1)^{m-1}UV^{(m)}$$

$$= U^{(m)}V + (-1)^{m-1}UV^{(m)}.$$

Hence, we get

$$U^{(m)}V = (-1)^m UV^{(m)} + \frac{d}{dt}\left[U^{(m-1)}V - U^{(m-2)}V^{(1)} + \ldots + (-1)^{(m-2)}U^{(1)}V^{(m-2)} + (-1)^{(m-1)}UV^{(m-1)}\right] \quad (1)$$

for $m = 0, 1, 2, \ldots, n$. Applying the formula (1) for

$U = u$, $V = a_0\bar{v}$, $m = n$

$U = u$, $V = a_1\bar{v}$, $m = n - 1$

$\vdots$

$U = u$, $V = a_{n-1}\bar{v}$, $m = 1$

$U = u$, $V = a_n\bar{v}$, $m = 0$, we get

$$\left(a_0\bar{v}\right) u^{(n)} = (-1)^n u\left(a_0\bar{v}\right)^{(n)} + \frac{d}{dt}\left[u^{(n-1)}a_0\bar{v} + \ldots + (-1)^{(n-1)}u(a_0\bar{v})^{(n-1)}\right]$$

$$\left(a_1\bar{v}\right) u^{(n-1)} = (-1)^{n-1} u\left(a_0\bar{v}\right)^{(n-1)} + \frac{d}{dt}\left[u^{(n-2)}a_1\bar{v} + \ldots + (-1)^{(n-2)}u(a_1\bar{v})^{(n-2)}\right]$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\left(a_{n-1}\bar{v}\right) u' = (-1)'u\left(a_{n-1}\bar{v}\right)' + \frac{d}{dt}\left[u\left(a_{n-1}\bar{v}\right)\right]$$

$$\left(a_n \bar{v}\right) u = u\ a_n \bar{v}$$

Adding all the above equations and *u*sing the definition of L_n and L_n^+, we get

$$\bar{v}\ L_n u = u\ \overline{L_n^+ v} + \frac{d}{dt}\left[\sum_{j=1}^{n} (-1)^{j-1}\ u^{(n-j)} \left(a_0 \bar{v}\right)^{(j-1)}\right] + \frac{d}{dt}\left[\sum_{j=1}^{n-1} (-1)^{j-1}\ u^{(n-1-j)} \left(a_0 \bar{v}\right)^{(j-1)}\right]$$

$$+\ldots+\frac{d}{dt}\left[\sum_{j=1}^{1} (-1)^{j-1}\ u^{(1-j)} \left(a_{n-1} \bar{v}\right)^{(j-1)}\right]$$

$$= u\ \overline{L_n^+ v} + \frac{d}{dt}\left[\sum_{m=1}^{n}\left\{\sum_{j=1}^{m} (-1)^{j-1}\ u^{(m-j)} \left(a_{n-m} \bar{v}\right)^{(j-1)}\right\}\right]$$

$$\Rightarrow \qquad \bar{v}\ L_n u - u\ \overline{L_n^+ v} = \frac{d}{dt}\ [P\ (u, v)]$$

where $P(u, v) = \sum_{m=1}^{n}\left[\sum_{j=1}^{m} (-1)^{j-1}\ u^{(m-j)} \left(a_{n-m} \bar{v}\right)^{(j-1)}\right]$

Note. $P(u, v)$ is called the bilinear concomitant associated with the differential operator L_n.

2.8.2. Corollary (Green's Formula).

If the a_n in the differential linear operator L_n and u, v are the same as in the above identity, then for any $t_1, t_2 \in I$.

$$\int_{t_1}^{t_2}\left(\bar{v}\ L_n u - u\ \overline{L_n^+ v}\right) dt = [P(u, v)]_{t=t_2} - [P(u, v)]_{t=t_1} = [P(u, v)]_{t_1}^{t_2}$$

where $P(u, v)$ is the bilinear concomitant associated with the differential operator L_n.

Proof. We know by Lagrange's identity that

$$\bar{v}\ L_n u - u\ \overline{L_n^+ v} = \frac{d}{dt}[P(u, v)]$$

Integrating both sides w.*r.t.* t between the limits t_1 and t_2, we get

$$\int_{t_1}^{t_2}\left(\bar{v}\ L_n u - u\ \overline{L_n^+ v}\right) dt = [P(u, v)]_{t_1}^{t_2} = [P(u, v)]_{t=t_2} - [P(u, v)]_{t=t_1}$$

Remark. If ψ is a non – trivial solution of the adjoint equation

$$\overline{L_n^+ v} = 0,\ t \in I \qquad (1)$$

then the problem of finding a non – trivial solution of the differential equation

$$L_n y = 0 \qquad (2)$$

is reduced to finding a solution ϕ on the interval I of an ordinary differential equation of order $(n-1)$, namely

$$P(y, \psi) = c \text{ (constant)}$$

2.9. The non homogeneous linear equation of order *n*. On a real t interval I, suppose $a_0 \neq 0$, $a_1, a_2, \ldots, a_n$ and b are continuous functions and consider the equation

$$L_n y = a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \ldots + a_n(t)\, y = b(t),$$

$t \in I$ which is defined to be the same as

$$y^{(n)} + \frac{a_1(t)}{a_0(t)}\, y^{(n-1)} + \ldots + \frac{a_n(t)}{a_0(t)}\, y = \frac{b(t)}{a_0(t)} \qquad (1)$$

This equation is called (in case $b(t) \neq 0$) a non – homogeneous linear equation of order n.

The system associated with this equation (1) is given by

$$\frac{d\hat{y}}{dt} = A(t)\,\hat{y} + \hat{b}(t),\ t \in I \qquad (2)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 1 & \cdots & 1 \\ \frac{-a_n}{a_0} & \frac{-a_{n-1}}{a_0} & \cdots & \cdots & \cdots & \frac{-a_1}{a_0} \end{bmatrix} \quad \hat{b}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{b(t)}{a_0(t)} \end{bmatrix}$$

2.9.1. Theorem. If $\phi_1, \phi_2, \ldots, \phi_n$ is a fundamental set of the homogeneous equation

$$L_n y = y^{(n)} + a_1(t)\, y^{(n-1)} + \ldots + a_n(t)\, y = 0 \ (a_k \in C \text{ on } I) \qquad (1)$$

then the solution ψ of the non – homogeneous equation

$$L_n\, y = b(t),\ b \in C \text{ on } I \qquad (2)$$

satisfying the initial condition $\psi(t_0) = \hat{y}_0$ is given by

$$\psi(t) = \psi_h(t) + \sum_{k=1}^{n} \phi_k(t) \left\{ \int_{t_0}^{t_1} \frac{W_k(\phi_1, \phi_2, \ldots, \phi_n)(s)}{W(\phi_1, \phi_2, \ldots, \phi_n)(s)} . b(s)ds \right\} \qquad (3)$$

where ψ_h is the solution of $L_n\, y = 0$ for which $\psi_h(t_0) = \hat{y}_0$ and $W_k(\phi_1, \phi_2, \ldots, \phi_n)$ is the determinant obtained from $W(\phi_1, \phi_2, \ldots, \phi_n)$ replacing the k^{th} column by $(0, \ldots, 0, 1)$.

Proof. First we recall a theorem that "If Φ is a fundamental matrix for linear homogeneous system $\frac{dy}{dt} = A(t)\,y$, then the function ϕ defined by $\phi(t) = \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\,b(s)\,ds$, $t \in I$ is a solution of non – homogeneous system $\frac{dy}{dt} = A(t)\,y + b(t)$ satisfying the initial condition $\phi(t_0) = 0$, $t_0 \in I$."

We know that system (2) is associated to the NH system given by

$$\frac{d\hat{y}}{dt} = A(t)\,\hat{y} + b(t),\ t \in I \qquad (*)$$

By above theorem the first compotent $\psi = \psi_1$ of the vector solution of (*) for which $\psi(t_0) = 0$ is given by

$$\psi(t) = \int_{t_0}^{t} \gamma_{1n}(t, s)\; b(s)\,ds$$

where $\gamma_{1n}(t, s)$ is the element in the first row and n[th] column of the matrix $\Phi(t)\Phi^{-1}(s)$. Recall that the element in the i[th] row and j[th] column of $\Phi(t)$ is $\phi_j^{(i-1)}$, and

$$\det\Phi(t) = W(\phi_1, \phi_2, \ldots, \phi_n)(t).$$

Now the element in the i[th] row and n[th] column of Φ^{-1} is given by

$$\frac{\tilde{\phi}_{in}}{W(\phi_1, \phi_2, \ldots, \phi_n)}$$

where $\tilde{\phi}_{in}$ is the cofactor of $\phi_i^{(n-1)}$ in Φ.

Therefore,

$$W(\phi_1, \phi_2, \ldots, \phi_n)(s)\; \gamma_{1n}(t, s) = \sum_{k=1}^{n} \phi_k(t)\; W_k(\phi_1, \phi_2, \ldots, \phi_n)(s)$$

where $W_k(\phi_1, \phi_2, \ldots, \phi_n)(s)$ is defined as in the statement of the theorem. Thus the solution ψ of $L_n\,y = b(t)$ satisfying $\psi(t_0) = 0$ is given by

$$\psi(t) = \sum_{k=1}^{n} \phi_k(t) \left\{ \int_{t_0}^{t_1} \frac{W_k(\phi_1, \phi_2, \ldots, \phi_n)(s)}{W(\phi_1, \phi_2, \ldots, \phi_n)(s)} . b(s)\,ds \right\}$$

and obviously (3) gives the solution satisfying $\psi(t_0) = \hat{y}_0$, if $\psi_h(t_0) = \hat{y}_0$.

2.10. The linear equation of order *n* with constant coefficient.

Consider the case where in L_n the functions $a_0 = 1, a_1,\ldots,a_n$ are all constants. Then I may be assumed to be the entire real t – axis. In this case,

$$L_n y = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0 \tag{1}$$

has its associated system

$$\frac{d\hat{y}}{dt} = A(t)\,\hat{y} \tag{2}$$

where A is the constant matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \tag{3}$$

2.10.1. Lemma. The characteristic polynomial for A in (3) is given by

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n \tag{*}$$

Proof. We shall prove the result by induction on n. For $n = 1$, we have $A = [-a_1]$ and hence

$\det(\lambda E_1 - A) = \lambda + a_1$, and therefore (*) is true for $n = 1$. Assume that result is true for $n - 1$. Then expand

$$\det(\lambda E_n - A) = \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & -1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & \lambda & -1 \\ a_n & a_{n-1} & a_{n-2} & & a_2 & \lambda + a_1 \end{vmatrix}$$

by the first column, we notice that the coefficient of λ is a determinant of order $(n - 1)$ which is equal to $\det(\lambda E_{n-1} - A_1)$, where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix}$$

So by induction hypothesis, we have

$$\det(\lambda E_{n-1} - A_1) = \lambda^{n-1} + a_1\lambda^{n-2} + \ldots + a_{n-1}.$$

The only other non-zero element in the first column is a_n, and the contribution to $\det(\lambda E - A)$ due to a_n is a_n, itself because the cofactor of a_n is 1. Hence

$$\det(\lambda E - A) = \lambda(\lambda^{n-1} + a_1\lambda^{n-2} + \ldots + a_{n-1}) + a_n.1 = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n.$$

Therefore characteristic polynomial for A is $f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$.

Note. The above lemma shows that $f(\lambda)$ can be obtained from $L_n y$ by changing $y^{(k)}$ to λ^k.

2.10.2. Theorem. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct roots of the characteristic equation

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \qquad (1)$$

and suppose λ_i has multiplicity m_i (i = 1, 2, …, s). Then a fundamental set for

$$L_n y = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0$$

is given by the n functions $t^k e^{t\lambda_i}$ $(k = 0, 1, 2, \ldots, m_i - 1;\ i = 1, 2, \ldots, s)$ (2)

Proof. We know that, if λ_I is a root of $f(\lambda) = 0$ with multiplicity m_i, then λ_i is also a root of the equations $f'(\lambda) = 0,\ f''(\lambda) = 0, \ldots,\ f^{(m_i-1)}(\lambda) = 0$.

It is clear that $L_n\left(e^{t\lambda}\right) = f(\lambda)e^{t\lambda}$ (3)

and, in general,

$$L_n\left(t^k\ e^{t\lambda}\right) = L_n\left(\frac{\partial^k}{\partial\lambda^k}e^{t\lambda}\right) = \frac{\partial^k}{\partial\lambda^k}\ L_n\left(e^{t\lambda}\right) = \frac{\partial^k}{\partial\lambda^k}\left(f(\lambda)\ e^{t\lambda}\right) \qquad \text{[By (3)]}$$

$$= \left[f^{(k)}(\lambda) + kf^{(k-1)}(\lambda)t + \frac{k(k-1)}{2!}f^{(k-2)}(\lambda)t^2 + \ldots + f(\lambda)t^k\right]e^{t\lambda}.$$

From this it is now obvious that, for any fixed i, $L_n\left(t^k\ e^{t\lambda_i}\right) = 0$ $(k = 0, 1, 2, \ldots, m_i - 1)$.

This proves that the functions in (2) are solutions of $L_n y = 0$. Now we shall prove that the functions in (2) are linearly independent. Let, if possible, functions in (2) are not linearly independent. Then there exists constants C_{ik} (not all zero) such that

$$\sum_{i=1}^{s}\sum_{k=0}^{m_i-1} C_{ik}\ t^k\ e^{t\lambda_i} = 0 \qquad \text{or} \qquad \sum_{i=1}^{\sigma} P_i\ (t)\ e^{t\lambda_i} = 0$$

where $P_i(t)$ are polynomial and $\sigma \le s$ is chosen so that $P_\sigma \not\equiv 0$ while $P_{\sigma+i}\ (t) \equiv 0$, $i \ge 1$. Divide above expression by $e^{t\lambda i}$ and differentiate enough times so that the polynomial $P_1(t)$ becomes zero. Note that the degrees and the non – identically vanishing nature of the polynomials multiplying $e^{(\lambda_i - \lambda_1)t}$, i > 1, do not change under this operation. Thus we get

$$\sum_{i=2}^{\sigma} \theta_i\ (t)\ e^{t\lambda_i} = 0$$

where $\theta_i(t)$ has the same degree as $P_i(t)$ for i $\geq$ 2. Repeating the procedure we get finally a polynomial $F(t)$ of degree equal to that of $P_\sigma(t)$ such that $F(t)$ = 0 for all t. This is impossible, since a polynomial can vanish only at isolated points. Thus the solutions are linearly independent.

2.11. Linear Second Order Equations. One of the most frequently occurring type of differential equation in mathematics and physical sciences is the linear second order differential equation of the form

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + q(t)\, u = h(t)$$

If $h(t)$ = 0, *t*hen differential equation is said to be homogenous. Unless otherwise stated, it is assumed that the functions $p(t) \neq 0$, $q(t)$ and $h(t)$ are continuous on the considered interval.

2.11.1. Theorem. If $u(t)$ and $v(t)$ are solutions of homogenous differential equation

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + q(t)\, u = 0 \qquad (1)$$

then there exist a constant c, depending on $u(t)$ and $v(t)$, such that their Wronskian W(t) satisfies

$$W(t) = \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} = \frac{c}{p(t)} \text{ that is, } u(t)\, v'(t) - u'(t)\, v(t) = \frac{c}{p(t)}$$

Proof. Since $u(t)$ and $v(t)$ are solutions of (1), so we must have

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + q(t)\, u(t) = 0 \qquad (2)$$

$$\frac{d}{dt}\left(p(t)\frac{dv}{dt}\right) + q(t)\, v(t) = 0 \qquad (3)$$

Multiplying equation (2) by $v(t)$ and equation (3) by $u(t)$ and subtracting, we obtain

$$v(t) \quad \frac{d}{dt}\left(p(t)\frac{du}{dt}\right) - u(t)\, \frac{d}{dt}\left(p(t)\frac{dv}{dt}\right) = 0 \qquad (4)$$

Consider

$$\frac{d}{dt}\Big[p(t)\big(u(t)v'(t)-u'(t)v(t)\big)\Big] = u(t)\frac{d}{dt}\big(p(t)v'(t)\big)+u'(t)p(t)v'(t)-v(t)\frac{d}{dt}\big(p(t)u'(t)\big)-v'(t)p(t)u'(t)$$

$$= u(t)\frac{d}{dt}\big(p(t)v'(t)\big)-v(t)\frac{d}{dt}\big(p(t)u'(t)\big) \qquad (5)$$

By (4) and (5), we obtain

$$\frac{d}{dt}\Big[p(t)\big(u(t)\; v'(t)-u'(t)\; v(t)\big)\Big] = 0 \Rightarrow p(t)\big(u(t)\; v'(t)-u'(t)\; v(t)\big) = c = \text{constant.}$$

$$\Rightarrow \qquad u(t)\ v'(t) - u'(t)\, v(t) = \frac{c}{p(t)}$$

2.11.2. Corollary. Show that the Wronskian of any pair of solutions of differential equation

$$\frac{d^2u}{dt^2} + q(t)u = 0 \text{ is a constant.}$$

Proof. Take $p(t) = 1$ in above theorem.

2.11.3. Lagrange's Identity. Consider the differential equations

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + q(t)\, u(t) = f(t) \text{ and } \frac{d}{dt}\left(p(t)\frac{dv}{dt}\right) + q(t)\, v(t) = g(t)$$

where $f(t)$ and $g(t)$ are continuous functions of t. Then, by the calculations employed in above theorem, we can obtain

$$\frac{d}{dt}\Big[p(t)\big(u(t)\ v'(t) - u'(t)\ v(t)\big)\Big] = g(t)\, u(t) - f(t)\, v(t).$$

This relations is known as Lagrange's Identity.

2.11.4. Green's Formula. Integrated form of Lagrange's Identity is known as Green's formula, that is,

$$\Big[p(s)\big(u(s)\ v'(s) - u'(s)\ v(s)\big)\Big]_a^t = \int_a^t \big(g(s)\, u(s) - f(s)\, v(s)\big)\, ds$$

where a, t are points of considered interval.

2.11.5. Theorem. If $u(t)$ is a non-zero solution of the homogeneous differential equation

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + q(t)\, u = 0,\ t \in I$$

Show that zeros of $u(t)$ cannot have a cluster point (limit point) in I.

Proof. Let, if possible zeros of $u(t)$ has a cluster point in I, say t_0 . Then by a theorem of real analysis, there exist a sequence of distinct points of interval I which converges t_0 the point t_0. Let $t \to t_0$ through the sequence of zeros of $u(t)$, then the continuity of $u(t)$ implies

$$\lim_{t \to t_0} u(t) = u(t_0) \Rightarrow u(t_0) = 0.$$

Now, by definition of derivative, we have $u'(t_0) = \lim_{t \to t_0} \dfrac{u(t) - u(t_0)}{t - t_0} = \lim_{t \to t_0} \dfrac{0 - 0}{t - t_0} = 0.$

So, we have obtained that $u(t_0) = u'(t_0) = 0$. Hence, by uniqueness theorem, we must have $u(t) = 0$, for all $t \in I$.

which is a contradiction because $u(t)$ is given to be a non-zero solution. So, our supposition is wrong and hence zero of $u(t)$ cannot have a cluster point in I.

2.11.6. Non-oscillatory Function. *A* real-valued function $f(t)$ defined and continuous in an interval $[a, b]$ is said to be non-oscillatory, if $f(t)$ has not more than one zero in $[a, b]$. If $f(t)$ has at least two zeros in $[a, b]$ then $f(t)$ is said to be an oscillatory function in $[a, b]$.

For example, (i) $f(t) = e^{-t} + e^{t}$ for $t \geq 0$ is non-oscillatory.

(ii) $f(t) = \sin t$, $t \in [0, 4\pi]$ then $f(t)$ is oscillatory.

2.11.7. Non-oscillatory Differential Equation. *A* second order differential equation

$$\frac{d^2u}{dt^2} + \left\{p(t)\frac{du}{dt}\right\} + q(t)\, u = h(t),\ t > 0$$

is called non-oscillatory, if every solution $u = u(t)$ of it is non-oscillatory, otherwise the given differential equation is called oscillatory.

2.11.8. Example. (i) $y'' + y = 0$, its general solution is, $y(t) = A\cos t + B\sin t$, $t \geq 0$

W.L.O.*G.* we can assume that both constants *A* and *B* are non-zero, otherwise $y(t)$ is trivially oscillatory. In that case (that is, $A \neq 0$, $B \neq 0$) the solution $y(t)$ has a zero at

$$t = n\pi + \tan^{-1}\frac{A}{B} \text{ for all } n = 0, 1, 2, \ldots$$

so this differential equation is oscillatory.

(ii) Consider the differential equation

$$y'' - y = 0 \text{ for all } t \geq 0.$$

Its general solution $y(t) = ae^{t} + be^{-t}$ where a, b are constants and this solution is non – oscillatory. Hence the given example is non-oscillatory.

2.11.9. Function Oscillate More Rapidly. Let $f(t)$ and $g(t)$ be two real valued and continuous functions defined in some closed interval $[a, b]$. Then the function $f(t)$ is said to oscillate more rapidly than $g(t)$ if the number of zeros of $f(t)$ in $[a, b]$ exceed the number of zeros of $g(t)$ in $[a, b]$ by more than one.

For example, let $f(t) = \sin t$, $t \in [0, 4\pi]$ and $g(t) = \sin 2t$, $t \in [0, 4\pi]$.

Here $g(t)$ oscillate more rapidly $f(t)$ as the number of zeros of g are half of the number of zeros of $f(t)$.

2.11.10. Strum Majorant and Strum Minorant. Consider the differential equation

$$\frac{d}{dt}\left(p_1(t)\frac{du}{dt}\right) + q_1(t)\, u = 0 \quad (1)$$

and
$$\frac{d}{dt}\left(p_2(t)\frac{du}{dt}\right) + q_2(t)\, u = 0 \quad (2)$$

where $p_i(t)$ and $q_i(t)$ are real valued continuous functions defined on interval J, such that

$$p_1(t) \geq p_2(t) \text{ and } q_1(t) \leq q_2(t) \quad (3)$$

Then the differential equation (2) is called **Strum majorant** for diff equation (1) on interval *J* and the differential equation (1) is known as **Strum minorant** for differential equation (2) on interval *J*.

Note. If, in addition $q_1(t) < q_2(t)$ or $p_1(t) > p_2(t) > 0$ and $q_2(t) \neq 0$, hold at some point *t* of the interval *J* then the differential equation (2) is called a strict Strum majorant for differential equation (1) on interval *J*.

2.11.11. Prufer Transformation. Consider the second order homogeneous linear differential equation

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + q(t)\,u = 0 \qquad (1)$$

on the interval *J* such that the coefficients $p(t)$ and $q(t)$ are real valued. Let $u = u(t)$ be a real valued solution of differential equation (1). In equation (1), we make the following substitution called Prufer substitution.

$$\left.p(t)\ u'(t) = \rho(t)\cos\phi(t),\ u(t) = \rho(t)\sin\phi(t)\ \right\} \qquad (2)$$

in which two dependent variables ρ and ϕ are defined by the formula

$$\left.\begin{aligned} \rho &= \sqrt{u^2 + p^2(u')^2} > 0 \\ \phi &= \tan^{-1}\left(\frac{u}{pu'}\right) \end{aligned}\right\} \qquad (3)$$

ρ is called the amplitude and ϕ the phase variable. Here u and u' cannot vanish simultaneously (that is, trivial solutions, which is not possible). So $\rho > 0$.

We now derive an equivalent system of differential equation for $\rho\ (t)$ and $\phi\,(t)$.

By equation (3),

$$\cot\phi = \frac{pu'}{u} \text{ (Since } p \neq 0) \qquad (4)$$

On differentiating w.*r.t.* '*t*' we get

$$\frac{d}{dt}(tan\phi) = \frac{d}{dt}\left(\frac{u}{pu'}\right) \Rightarrow \sec^2\phi\ \frac{d\phi}{dt} = \frac{p(t)\big(u'(t)\big)^2 - u(t)\dfrac{d}{dt}\big(p(t)\,u'(t)\big)}{\left[p(t)\,u'(t)\right]^2}$$

$$= \frac{1}{p(t)} + \frac{u(t)\cdot q(t)u(t)}{\left[p(t)\,u'(t)\right]^2} \qquad \text{[By (1)]}$$

$$= \frac{1}{p(t)} + \tan^2\phi\cdot q(t) \qquad \text{[By (3)]}$$

$$\Rightarrow \quad \frac{d\phi}{dt} = \frac{1}{p(t)}\cos^2\phi + q(t)\sin^2\phi \qquad (5)$$

Also, $\rho^2 = u^2 + p^2 (u')^2$ (6)

Differentiating equation (6), we get

$$2\rho \frac{d\rho}{dt} = 2uu' + 2(pu') \frac{d}{dt}(pu') = 2uu' + 2(pu')(-qu)$$

$$= 2.\frac{1}{p(t)} \rho^2 \sin\phi \cos\phi + 2\rho\cos\phi(-qt\ \rho\ \sin\phi) = \left[\frac{1}{p(t)} - q(t)\right] \rho^2 \sin^2\phi$$

$$\Rightarrow \quad \frac{d\rho}{dt} = \frac{1}{2}\left[\frac{1}{p(t)} - q(t)\right] \rho\ \sin 2\phi \qquad (7)$$

The system consisting of first order differential equation (5) and (7) is equivalent to the second order differential equation (1). This system is called Prufer system associated with the differential equation (1).

The differential equation (5) of the Prufer system is a first order differential equation in $\phi(t)$ alone and not containing the other dependent variable ρ. If a solution $\phi = \phi(t)$ of differential equation (5) is known, *t*hen a corresponding solution of first order ordinary differential equation (7) is obtained by integration.

Remark. Each solution of the Prufer system depend *u*pon two constants, *t*he initial amplitude $\rho_0 = \rho(t_0)$ and the initial phase $\phi = \phi(t_0)$. Changing the constant ρ_0 just multiplies a solution $u(x)$ of differential equation (1) by a constant factor. Thus the zeros of any non-trivial solution $u = u(t)$ of differential equation (1) can be located by studying only the first order differential equation (5).

2.11.12. Theorem. Let $u_1(t)$ and $u_2(t)$ be two linearly independent solutions of differential equation.

$$\frac{d}{dt}\left\{p(t)\frac{du}{dt}\right\} + q(t)\,u = 0 \qquad (1)$$

in $[a, b]$ with $p(t) > 0$. Prove that $u_1(t)$ and $u_2(t)$ do not admit common zeros.

Proof. Let, if possible, *t*hey have a common zero at $t = t_0$, say, for some $t_0 \in [a, b]$.

Then $u_1(t_0) = u_2(t_0) = 0$

By Abel's lemma, we have

$$p(t)\left[u_1(t)\,u_2'(t) - u_1'(t)\,u_2(t)\right] = c \qquad (2)$$

for all $t \in [a, b]$, where c is some constant.

To find value of c, we put $t = t_0$ in equation (2), and we get $c = 0$ [Since $u_1(t_0) = u_2(t_0) = 0$].

Using $c = 0$ in equation (2), we get

$$p(t)\left[u_1(t)\,u_2'(t) - u_1'(t)\,u_2(t)\right] = 0 \text{ for all } t \in [a, b]$$

$$\Rightarrow \quad u_1(t)\,u_2'(t) - u_1'(t)\,u_2(t) = 0 \qquad \Rightarrow W(u_1, u_2)(t) = 0 \text{ for all } t \in [a, b]$$

Then solutions $u_1(t)$ and $u_2(t)$ are linear dependent, which is a contradiction.

Hence $u_1(t)$ and $u_2(t)$ cannot have common zeros.

2.11.13. Theorem. Let $u(t)$ be a non – trivial solution of differential equation

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + q(t)\, u = 0 \text{ in } [a, b] \qquad (1)$$

Prove that zeros of $u(t)$ are isolated.

Proof. Let $t = t_0$ be a zero of $u(t)$ then $u(t_0) = 0$. Since u is a non-trivial solution of differential equation (1) it follows that $u'(t_0) \neq 0$. Now there are two possibilities**.-**

Case(i). When $u'(t_0) > 0$. Since the derivative of u is continuous and is positive at $t = t_0$, it follows that $u(t)$ is increasing function and it has a positive neighbourhood that is, $u(t)$ is non-zero in some neighbourhood of t_0.

Case (ii). When $u'(t_0) < 0$. In this case, $u(t)$ is a decreasing function and it has no zero in some neighbourhood of $t = t_0$. Hence the zeros of $u(t)$ are isolated.

2.11.14. Theorem. Let $u_1(t)$ and $u_2(t)$ be non-trivial linearly dependent solution of differential equation

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + q(t)\, u = 0 \qquad (1)$$

on $[a, b]$ and $p(t) > 0$. Then the zeros of $u_1(t)$ and $u_2(t)$ are identically same.

Proof. Since $u_1(t)$ and $u_2(t)$ are linearly dependent on $[a, b]$ so there exist constants c_1 and c_2

(not both zero) such that

$$c_1\, u_1(t) + c_2\, u_2(t) = 0 \text{ for all } t \in [a, b]. \qquad (2)$$

Now, we shall prove that c_1 and c_2 are both non-zero. If $c_2 = 0$, then (2) gives

$$c_1 u_1(t) = 0 \text{ for all } t \in [a, b] \Rightarrow c_1 = 0$$

since $u(t)$ is non – trivial solution. This is a contradiction to the assumption that c_1 and c_2 are not both zero. So, we must have $c_2 \neq 0$.

Similarly, $c_1 \neq 0$.

Now, let $t = t_0$ be a zero of $u_1(t)$, then $u_1(t_0) = 0$ and equation (2) gives,

$$c_2 u_2(t_0) = 0 \Rightarrow u_2(t_0) = 0 \; [\because c_2 \neq 0\,]$$

$\Rightarrow$ t_0 is also a zero of $u_2(t)$.

Thus every zero of $u_1(t)$ is also a zero of $u_2(t)$. Similarly every zero of $u_2(t)$ is also a zero of $u_1(t)$. Hence $u_1(t)$ and $u_2(t)$ both have same zeros.

2.11.15. Example. Consider the differential equation $\frac{d^2u}{dt^2} + u = 0$ that is, $p(t) \equiv 1$, $q(t) \equiv 1$.

Let $u_1(t) = A \sin t$, $u_2(t) = B \sin t$ where A and B are arbitrary constants. Then $u_1(t)$ and $u_2(t)$, two non-trivial linearly dependent solutions of the given differential equation have the common zeros at

$t = \pm\, n\pi$, $n = 0, 1, 2, \ldots$ and no other zero.

2.11.16. Theorem. Let $u_1(t)$ and $u_2(t)$ be two non-trivial solutions of differential equation

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + q(t)u(t) = 0 \text{ on } [a, b] \qquad (1)$$

with $p(t) > 0$. If $u_1(t)$ and $u_2(t)$ have common zeros on $[a, b]$, then they are linearly dependent on $[a, b]$.

Proof. We know by Abel's lemma, that

$$p(t)\,[u_1(t)\; u_2'(t) - u_1'(t)\, u_2(t)] = c \text{ (constant)} \qquad (2)$$

on $[a, b]$. Let $t = t_0$ be a common zero of $u_1(t)$ and $u_2(t)$. Then $u_1(t_0) = u_2(t_0) = 0$

Using these value in (2), we get $c = 0$. (3)

As $p(t) > 0$, equation (1) and equation (3) gives

$$u_1(t)\; u_2'(t) - u_1'(t)\, u_2(t) = 0 \Rightarrow W(u_1, u_2)(t) = 0 \text{ for all } t \in [a, b].$$

This implies that the solutions $u_1(t)$ and $u_2(t)$ are linearly dependent on $[a, b]$. Hence proved.

2.11.17. Theorem. If $u_1(t)$ and $u_2(t)$ be two real valued non-trivial linearly independent solutions of differential equation $\frac{d}{dt}\left(p(t)\frac{du}{dt}\right) + q(t)\, u(t) = 0$ on $[a, b]$ with $p(t) > 0$. Then the zeros of $u_1(t)$ separate and are separated by those of $u_2(t)$.

Proof. Let $t = t_1, t_2$ be two consecutive zeros of $u_1(t)$ on $[a, b]$ so $u_1(t_1) = u_2(t_2) = 0$

Since $u_1(t)$ and $u_2(t)$ are linearly independent on $[a, b]$ so they do not admit common zeros. So, we must have $u_2(t_1) \neq 0$ and $u_2(t_2) \neq 0$.

We shall now show that u_2 has one zero in open interval (t_1, t_2). Let, if possible, that it does not happen that is, $u_2(t)$ has no zero in the open interval (t_1, t_2).

Then the quotient function $\left(\frac{u_1}{u_2}\right)(t)$ satisfies all conditions of Rolle's theorem.

Therefore by Rolle's theorem, there exist a point $c \in (t_1, t_2)$ such that

$$\frac{d}{dt}\left(\frac{u_1}{u_2}\right)(c) = 0$$

$$\Rightarrow \quad \frac{u_1(c)\, u_2'(c) - u_1'(c)\, u_2(c)}{\left(u_2(c)\right)^2} = 0 \qquad \Rightarrow \quad u_1(c)\, u_2'(c) - u_1'(c)\, u_2(c) = 0$$

$$\Rightarrow \quad W(u_1, u_2)(c) = 0$$

which is a contradiction, because the solutions $u_1(t)$ and $u_2(t)$ are linearly independent so their Wronskian must be non – zero at every point of interval [a, b].

Hence our supposition is wrong therefore $u_2(t)$ has at least one zero in the interval (t_1, t_2). This proves that the zeros of $u_1(t)$ are separated by zeros of $u_2(t)$. Now we shall prove that $u_2(t)$ has exactly one zero in the open interval (t_1, t_2).

Let, if possible, $u_2(t)$ has two consecutive zeros $t_3 < t_4$ in the open interval (t_1, t_2).

On interchanging the roles of solutions of $u_1(t)$ and $u_2(t)$ in the above proved conjecture, we get that there is at least one zero, say $t = t_5$, of the solution $u_1(t)$ in the open interval $(t_3, t_4) \subset (t_1, t_2)$. This is a contradiction to the assumption that t_3 and t_4 are two consecutive zeros.

2.11.18. Sturm Fundamental Comparison Theorem. Consider the differential equations

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right)+q_1(t)u = 0 \tag{1}$$

and
$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right)+q_2(t)u = 0 \tag{2}$$

on the interval [a, b] such that $p(t) > 0$ have a continuous derivative on $[a, b]$ and $q_1(t) < q_2(t)$ be continuous on [a, b]. Let $u_1(t)$ and $u_2(t)$ be non – trivial solutions of equation (1) and equation (2) respectively. Prove that between any two consecutive zeros of $u_1(t)$ on $[a, b]$, there lies at least one zero of $u_2(t)$.

Proof. Let $t_1 < t_2$ be two consecutive zeros of $u_1(t)$ on $[a, b]$. Then by hypothesis, we have

$$u_1(t_1) = u_2(t_2) = 0$$

Now $u_1(t)$ and $u_2(t)$ are solutions of (1) and (2) respectively, so

$$\frac{d}{dt}\left(p(t)\frac{du_1}{dt}\right)+q_1(t)\ u_1(t) = 0 \tag{3}$$

for all $t \in [a, b]$

and
$$\frac{d}{dt}\left(p(t)\frac{du_2}{dt}\right)+q_2(t)\ u_2(t) = 0 \tag{4}$$

Multiplying equation (3) by $u_2(t)$ and equation (4) by $u_1(t)$ and subtracting, we get

$$u_2(t)\left[\frac{d}{dt}\left(p(t)\frac{du_1}{dt}\right)+q_1(t)\ u_1(t)\right] - u_1(t)\left[\frac{d}{dt}\left(p(t)\frac{du_2}{dt}\right)+q_2(t)\ u_2(t)\right] = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left[p(t)\left(u_1'(t)\ u_2(t)-u_1(t)\ u_2'(t)\right)\right] + q_1(t)\ u_1(t)\ u_2(t) - q_2(t)\ u_1(t)\ u_2(t) = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left[p(t)\left(u_1'(t)\ u_2(t)-u_1(t)\ u_2'(t)\right)\right] = [q_2(t) - q_1(t)]\ u_1(t)\ u_2(t) \tag{6}$$

Integrating both sides w.$r.t.$ 't' over $[t_1, t_2]$,]

$$\left[p(t)\{u_1'(t)u_2(t)-u_1(t)u_2'(t)\}\right]_{t_1}^{t_2} = \int_{t_1}^{t_2}\left(q_2(t)-q_1(t)\right) u_1(t)\, u_2(t)\, dt$$

Now using $u_1(t_1) = u_2(t_2) = 0$ in L.H.S. of above, we get

$$p(t_2)\ u_1'(t_2)\,u_2(t_2) - p(t_1)\ u_1'(t_1)\,u_2(t_1) = \int_{t_1}^{t_2}\left(q_2(t)-q_1(t)\right) u_1(t)\, u_2(t)\, dt \qquad (7)$$

Let, if possible, we assume that $u_2(t)$ does not have any zero in the open interval (t_1, t_2).

Then W.L.O.*G*., we can assume that $u_1(t) > 0$ and $u_2(t) > 0$ in (t_1, t_2).

Since $u_1(t_2) = 0$ and $u_1(t) = 0$ for all t in (t_1, t_2), therefore we have $u_1'(t_2) < 0$. (8)

Since $u_2(t) > 0$ on (t_1, t_2), therefore $u_2(t_2) \geq 0$. (9)

Also by hypothesis $p(t_2) > 0$. (10)

By (8), (9) and (10), we get

$$p(t_2)\ u_1'(t_2)\,u_2(t_2) \leq 0 \qquad (11)$$

Similarly, we can show that

$$p(t_1)\ u_1'(t_1)\,u_2(t_1) \geq 0 \qquad (12)$$

By (11) and (12), we have that L.H.S. of equation (7) is not positive that is,

$$p(t_2)\ u_1'(t_2)\,u_2(t_2) - p(t_1)\ u_1'(t_1)\,u_2(t_1) \leq 0$$

$$\Rightarrow \quad \int_{t_1}^{t_2}\left(q_2(t)-q_1(t)\right) u_1(t)\, u_2(t)\, dt \leq 0 \qquad (13)$$

But by hypothesis, $q_2(t) - q_1(t) > 0$ on (t_1, t_2) and also $u_1(t) > 0$, $u_2(t) > 0$ on (t_1, t_2).

So we must have,

$$\int_{t_1}^{t_2}\left(q_2(t)-q_1(t)\right) u_1(t)\, u_2(t) > 0 \qquad (14)$$

which is a contradiction, as (13) and (14) contradict each other.

Hence $u_2(t)$ must have a zero between two consecutive zeros of $u_1(t)$.

This completes the proof.

2.11.19. Example. Verify the Sturm's fundamental comparison theorem in the case of real solutions of the differential equations

$$\frac{d^2u}{dt^2} + A^2u = 0 \qquad (1)$$

and
$$\frac{d^2u}{dt^2} + B^2u = 0 \qquad (2)$$

where A and B are constants such that $B > A > 0$.

Solution. Let $u_1(t) = \sin At$ and $u_2(t) = \sin Bt$. Then $u_1(t)$ and $u_2(t)$ are real solutions of differential equations (1) and (2) respectively.

Consecutive zeros of $u_1(t)$ are

$$\frac{n\pi}{A}, \frac{(n+1)\pi}{A} \text{ for } n = 0, \pm 1, \pm 2, \ldots$$

By Sturm's comparison theorem (with $p(t) \equiv 1$, $q_1(t) = A^2$, $q_2(t) = B^2$, $q_2 > q_1$) the solution $u_2(t)$ of differential equation (2) has at least one zero, say, ξ_n, between the zeros $\frac{n\pi}{A}$ and $\frac{(n+1)\pi}{A}$ of $u_1(t)$ that is,

$$\frac{n\pi}{A} < \xi_n \; \frac{(n+1)\pi}{A}, n = 0, \pm 1, \pm 2, \ldots$$

We may take $\xi_n = \frac{(n+1)\pi}{B}$ as zero of $u_2(t)$.

In particular for $n = 0$

$t_1 = 0$, $t_2 = \frac{\pi}{A}$ are two consecutive zeros of $u_1(t)$. The zero $t = \frac{\pi}{B}$ of $u_2(t)$ lies between t_1 and t_2 as

$0 < \frac{\pi}{B} < \frac{\pi}{A}$ [Since $B > A > 0$]. Hence the verification.

2.11.20. Theorem. Consider the differential equations

$$\frac{d}{dt}\left\{p(t)\frac{du}{dt}\right\} + q_1(t)u = 0 \qquad (1)$$

$$\frac{d}{dt}\left\{p(t)\frac{dv}{dt}\right\} + q_2(t)v = 0 \qquad (2)$$

where $p(t) > 0$, $q_2(t) > q_1(t)$ on $a \le t \le b$. Further more either

(i) $\frac{u'(a)}{u(a)} \ge \frac{v'(a)}{v(a)}$, $u(a) \ne 0$, $v(a) \ne 0$

or (ii) $u(a) = 0$, $v(a) = 0$

Then $v(t)$ has atleast as many zeros in $[a, b]$ as $u(t)$. In case (i), if the zeros of $u(t)$ are $t_1, t_2, \ldots, t_n$ with $a < t_1 < t_2 < \ldots < t_n \le b$ and the zeros of $v(t)$ are $\xi_1, \xi_2, \ldots, \xi_m$ with $a < \xi_1 < \xi_2 < \ldots < \xi_m \le b$ then prove that $\xi_k < t_k$.

Proof. By the fundamental comparison theorem, we know that, "Between any two zeros of $u(t)$, there is at least one zero of $v(t)$".

Thus, $v(t)$ has atleast $n-1$ zeros in $[a, b]$. It is sufficient to show that $v(t)$ has a zero lying in the interval $[a, t_1]$. In case (ii), this is obvious as $t = a$ is also a zero of $v(t)$. In case (i), we assume without loss of generality that $u(t) > 0$ and $v(t) > 0$ in open interval (a, t_1).

Multiplying equation (1) by $v(t)$, (2) by $u(t)$ and subtracting, we get

$$\frac{d}{dt}\left[p(t)\left(u'(t)\,v(t) - u(t)\,v'(t)\right)\right] = \left(q_2(t)-q_1(t)\right)\,u(t)\,v(t)$$

Integrating both sides over (a, t_1), we get

$$\left[p(t)\{u'(t)\,v(t) - u(t)\,v'(t)\}\right]_a^{t_1} = \int_a^{t_1}\left(q_2(t)-q_1(t)\right)\,u(t)\,v(t)\,dt \qquad (*)$$

L.H.S. of (*) $= p(t_1)\;u'(t_1)v(t_1) - p(a)\left[u'(a)v(a)-u(a)\,v'(a)\right]$ [Since $u(t_1) = 0$]

$$= p(t_1)u'(t_1)v(t) - p(a)\left\{\frac{u'(a)\,v(a) - u(a)\,v'(a)}{u(a)\,v(a)}\right\}.u(a)v(a)$$

$$= p(t_1)\;u'(t_1)v(t_1) - p(a)\left\{\frac{u'(a)}{u(a)}-\frac{v'(a)}{v(a)}\right\}u(a)\;v(a) \le 0 \qquad (3)$$

$$\left[Since\ \frac{u'(a)}{u(a)} \ge \frac{v'(a)}{v(a)},\ p(a)\ge 0,\ p(t_1) \ge 0,\ u'(t_1)<0, v(t_1)>0,\ u(a)>0,\ v(a)>0\right]$$

But R.H.S. of (*) $= \int_a^{t_1}\left(q_2(t)-q_1(t)\right)\,u(t)\,v(t)\,dt > 0$ (4)

which is contradiction to (*) since (3) and (4) contradicting each other. Hence $v(t)$ must have a zero lying in the interval (a, t_1).

2.12. Check Your Progress.

1. Consider the differential equation

$$\frac{d}{dt}\left\{p(t)\frac{du}{dt}\right\}+q_1(t)u = 0 \qquad (1)$$

$$\frac{d}{dt}\left\{p(t)\frac{dv}{dt}\right\}+q_2(t)v = 0 \qquad (2)$$

where $p(t) > 0$, $q_2(t) > q_1(t)$ on $[a, b]$. Further let either

(i) $\dfrac{u'(a)}{u(a)} \ge \dfrac{v'(a)}{v(a)}$, $u(a) \neq 0$, $v(a) \neq 0$

or (ii) $u(a) = 0$, $v(a) = 0$

Suppose that $u(t)$ and $v(t)$ have the same number of zeros in $[a, b]$. Then show that

$$\frac{u'(b)}{u(b)} > \frac{v'(b)}{v(b)} \text{ if } u(b) \neq 0.$$

Answer.

Since $u(b) \neq 0$ and $v(t)$ has as many zeros in $[a, b]$ as $u(t)$, it follows that $v(b) \neq 0$, since, in this case, $t_n < b$ and $v(t)$ has atleast as many zeros as $u(t)$ in $[a, b]$ by virtue of Sturm comparison theorem. Multiplying equation (1) by $v(t)$ and (2) by $u(t)$, subtracting and then integrating over (t_n, b), we get

$$\left[p(t)\{u'(t)\, v(t) - u(t)\, v'(t)\}\right]_{t_n}^{b} = \int_{t_n}^{b} \left(q_2(t) - q_1(t)\right) u(t)\, v(t)\, dt \qquad (3)$$

W.L.O.G. We may assume that $u(t) > 0$, $v(t) > 0$ in $[t_n, b]$

Then, $$\int_{t_n}^{b} \left(q_2(t) - q_1(t)\right) u(t)\, v(t)\, dt > 0 \qquad (4)$$

By (3) and (4), we get $\left[p(t)\{u'(t)\, v(t) - u(t)\, v'(t)\}\right]_{t_n}^{b} > 0$

$$\Rightarrow \quad p(b)\, u'(b) v(b) - p(b)\, u(b)\, v'(b) - p(t_n)\, u'(t_n)\, v(t_n) + p(t_n)\, u(t_n)\, v'(t_n) > 0$$

$$\Rightarrow \quad p(b)\left[u'(b)v(b) - u(b)v'(b)\right] > p(t_n)\, u'(t_n)\, v(t_n) \quad [u(t_n) = 0]$$

But $p(t_n)\, u'(t_n)\, v(t_n) > 0$ [Since $p(t_n) > 0$, $v(t_n) > 0$, $v(t_n) > 0$, $u'(t_n) > 0$]

Therefore, $u'(b)\, v(b) - u(b)\, v'(b) > 0 \Rightarrow \dfrac{u'(b)}{u(b)} > \dfrac{v'(b)}{v(b)}$.

2.13. Summary.

In this chapter, we discussed various properties of solutions of linear homogeneous systems. Also, by using the fundamental matrix of a linear homogeneous system we can obtain a solution of linear non-homogeneous system.

Books Suggested:

1. Ross, S.L., Differential equations, John Wiley and Sons Inc., New York, 1984.
2. Boyce, W.E., Diprima, R.C., Elementary differential equations and boundary value problems, John Wiley and Sons, Inc., New York, 4th edition, 1986.
3. Simmon, G.F., Differential Equations, Tata McGraw Hill, New Delhi, 1993.

3

Autonomous System

Structure

3.1. Introduction. This chapter includes the results related to autonomous system, critical points, paths approaching and entering a critical point. The behavior of a path at a critical point is of great interest of study.

3.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Stability and asymptotic stability of a critical point.

(ii) Various types of critical points.

(iii) Almost linear syetems.

3.1.2. Keywords. Critical Point, Node, Saddle Point, Center, Spiral Point, Characteristic Equation.

3.2. Autonomous System. Consider the system

$$\frac{dx}{dt} = F(x, y) \qquad \frac{dy}{dt} = G(x, y) \qquad (1)$$

where F and G are continuous and have continuous first partial derivative throughout the xy plane. A system of this kind in which the independent variable t does not appear in the function F and G is called an autonomous system.

3.2.1. Example. Consider the second order differential equation

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right) \qquad (2)$$

If we imagine a simple dynamical system consisting of a particle of unit mass (m = 1) moving on the t – axis and if $f\left(x, \frac{dx}{dt}\right)$ is the force acting on it, then equation (2) is the equation of motion.

The values of x (position) and $\frac{dx}{dt}$ (velocity), which at each instant characterize the state of the system, are called its phases. The plane of the variables x and $\frac{dx}{dt}$ is called the phase plane.

If we introduce the variable

$$y = \frac{dx}{dt} \qquad (3)$$

then the second order differential equation (2) is equivalent to the following system

$$\frac{dx}{dt} = y \quad \frac{dy}{dt} = f(x, y) \qquad (4)$$

which is an autonomous system.

3.2.2. Solution and path of an autonomous system. Consider the autonomous system

$$\frac{dx}{dt} = F(x, y) \qquad \frac{dy}{dt} = G(x, y) \qquad (1)$$

By a theorem of system of differential equations, we have that for any given real number t_0 and any pair (x_0, y_0) of real numbers, there exists a unique solution

$$x = x(t),\ y = y(t) \qquad (2)$$

of the system (1) such that $x(t_0) = x_0$ and $y(t_0) = y_0$. If $x(t)$ and $y(t)$ are not both constant functions, then system (2) represents a curve in xy - plane which is called path (or orbit or trajectory) of the system (1).

If the ordered pair of functions defined by (2) is a solution of (1) and t, is any real number, then it is easy to see that the ordered pair of functions defined by

$$x = x(t - t_1),\ y = y(t - t_1) \qquad (3)$$

is also solution of (1). Assuming that x and y in (2) are not both constant functions and $t_1 \neq 0$, the solutions defined by (2) and (3) are two different solutions of (1). However, these two different solutions are simply the different parameterizations of the same path i.e. shifting of the origin from $t = 0$ to $t = t_1$. We thus observe that the terms solution and path are not synonymous. On the one hand, a solution of (1) is an ordered pair of functions (x, y) such that $x = x(t)$, $y = y(t)$ simultaneously satisfy the two equations of the system (1) identically ; on the other hand a path of (1) is a curve in the xy – phase plane, which may be defined parametrically by more than one solution of (1).

A path is a directed curve. The direction of increasing t along a given path is same for all solutions representing the path in our figure. We shall use arrows to indicate the direction in which path is traced out as t increases.

3.3. Critical point. Given the autonomous system

$$\frac{dx}{dt} = F(x, y) \text{ and } \frac{dy}{dt} = G(x, y) \qquad (1)$$

a point (x_0, y_0) at which both $F(x_0, y_0) = 0$ and $G(x_0, y_0) = 0$ is called a critical point of (1).

A critical point is also called equilibrium point or singular point.

3.3.1. Example. Consider the linear autonomous system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x \qquad (1)$$

The general solution of this system may be put as

$$x = c_1 \sin t - c_2 \cos t \quad , y = c_1 \cos t + c_2 \sin t$$

where c_1 and c_2 are arbitrary constants. The solution satisfying $x(0) = 0$ and $y(0) = 1$ is clearly

$$x = \sin t, y = \cos t \qquad (2)$$

This solution defines a path C_1 in the xy – plane. The solution satisfying the condition $x(0) = –1$ and $y(0) = 0$ is clearly

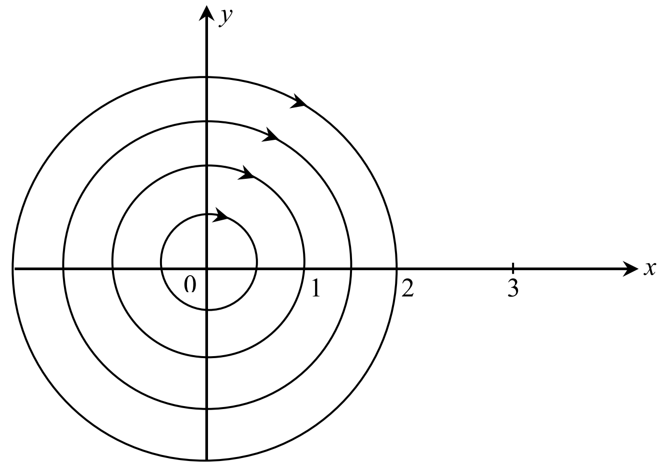
$$x = \sin\left(t - \frac{\pi}{2}\right), \qquad y = \cos\left(t - \frac{\pi}{2}\right) \qquad (3)$$

The solution (3) is different from the solution (2), but (3) also defines the same path C_1. That is, the ordered pair of functions defined by (2) and (3) are two different solutions of (1) which are different parameterizations of the same path C_1. Eliminating t from either (2) or (3) we obtain the equation $x^2 + y^2 = 1$ of the path C_1 in the xy, phase plane. Thus the path C_1 is the circle with centre at (0, 0) and radius 1.From either (2) or (3) we see that the direction associated with C_1 is the clockwise direction. Eliminating t between the equations of system (1) we obtain the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \qquad (4)$$

which gives the slope of the tangent to the path of (1) passing through the point (x, y), provided

$(x, y) \neq (0, 0)$. The one parameter family of solutions $x^2 + y^2 = c^2$ of equation (4) gives the one – parameter family of paths in the xy phase plane. Several of these are shown in the figure below. The path C_1 referred above is, of course, that for which $c = 1$.



By equation (1), i.e. comparing with $\frac{dx}{dt} = F(x, y)$

$\frac{dy}{dt} = G(x, y)$

We see that $F(x, y) = y$ and $G(x, y) = -x$. Therefore the only critical point of the system is the origin (0, 0). Given any real number t_0, the solution $x = x(t)$, $y = y(t)$ such that $x(t_0) = 0 = y(t_0)$ is simply $x = 0$, $y = 0$, for all t.

3.3.2. Isolated critical point. A critical point (x_0, y_0) of the autonomous system

$$\frac{dx}{dt} = F(x, y) \quad \frac{dy}{dt} = G(x, y) \qquad (1)$$

is called an isolated critical point if there exist a circle $(x-x_0)^2 + (y-y_0)^2 = r^2$ about the point (x_0, y_0) such that (x_0, y_0) is the only critical point of (1) within this circle.

Note. For convenience, we shall take the critical point (x_0, y_0) to be the origin (0, 0). There is no loss in generality in doing so, for if $(x_0, y_0) \neq (0, 0)$, then the translation of coordinates

$$\xi = x - x_0, \quad \eta = y - y_0$$

transforms (x_0, y_0) into the origin in $\xi\ \eta$ plane.

3.3.3. Path *A*pproaching Critical Point. Let $x = x(t)$, $y = y(t)$ is a solution which parametrically represents the path C, and let (0, 0) be a critical point of the autonomous system $\frac{dx}{dt} = F(x, y)$ $\frac{dy}{dt} = G(x, y)$.

Then we say that the path C approaches the critical point (0, 0) as $t \to \infty$ if

$$\lim_{t\to\infty} x(t) = 0 \text{ and } \lim_{t\to\infty} y(t) = 0.$$

In like manner, a path C_1 approaches the critical point (0, 0) as $t \to -\infty$ if

$$\lim_{t\to-\infty} x_1(t) = 0 \text{ and } \lim_{t\to-\infty} y_1(t) = 0$$

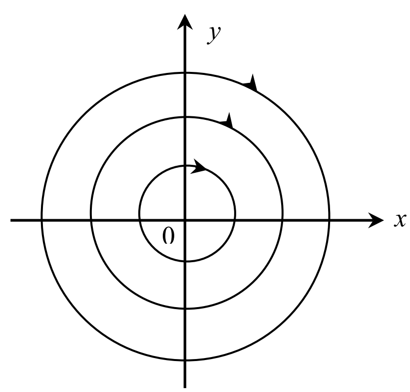
where $x = x_1(t)$ and $y = y_1(t)$ is a solution defining the path C_1.

3.3.4. Path Entering Critical Point. Let $x = x(t)$ and $y = y(t)$ be a solution which parametrically represents the path C and let (0, 0) be the critical point of the autonomous system $\frac{dx}{dt} = F(x, y)$, $\frac{dy}{dt} = G(x, y)$ to which C approaches as $t \to \infty$. Then we say that C enters the critical point (0, 0), as $t \to \infty$ if $\lim_{t\to\infty} \frac{y(t)}{x(t)}$ exists or if this quotient becomes either positively or negatively infinite as $t \to \infty$.

We observe that the quotient $\frac{y(t)}{x(t)}$ represents the slope of the line joining critical point (0, 0) and a point R with coordinates $(x(t), y(t))$ on C. Thus when we say that a path C enters the critical point (0, 0) as $t \to \infty$ we mean that the line joining (0, 0) and a point R tracing out C approaches a definite 'limiting' direction as $t \to \infty$.

3.4. Types of critical points.

3.4.1. Center. The isolated critical point (0, 0) of autonomous system is called a center if there exists a neighbourhood of (0, 0) which contains a countably infinite number of closed path C_n ($n = 1, 2, \ldots$), each of which contains (0, 0) in its interior, and which are such that the diameters of the paths approach 0 as $n \to \infty$, but (0, 0) is not approached by any path either as $t \to \infty$ or as $t \to -\infty$.



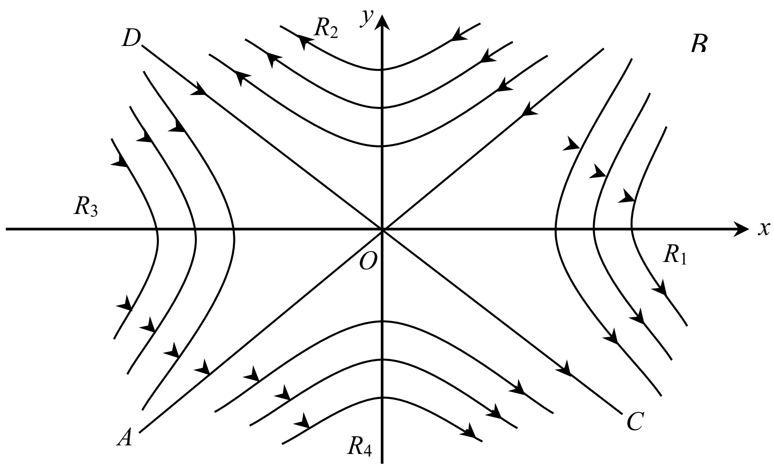
The critical point (0, 0) of adjoining figure is called a center. Such a point is surrounded by an infinite family of closed paths, members of which are arbitrarily close to (0, 0), but it is not approached by any path either as $t \to \infty$ or as $t \to -\infty$.

3.4.2.. Saddle Point. The isolated critical point (0, 0) is called a saddle point if there exist a neighbourhood of (0, 0) in which the following two conditions hold.

(i) There exist two paths which approach and enter (0, 0) from a pair of opposite directions as $t \to \infty$, and there exist two paths which approach and enter (0, 0) from a different pair of opposite directions as $t \to -\infty$.

(ii) In each of the four domains between any two of the four directions in (i) there are infinitely many paths which are arbitrarily close to (0, 0) but which do not approach (0, 0) either as $t \to \infty$ or as $t \to -\infty$.

The critical point (0, 0) of adjoining figure is a saddle point which is such that



(i) It is approached and entered by two half-line paths (A_0 and B_0) as $t \to +\infty$, these two paths forming the geometric curve AB.

(ii) It is approached and entered by two half – line path (C_0 and D_0) as $t \to -\infty$, these two paths forming the geometric curve CD.

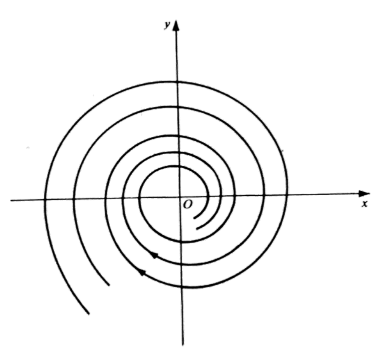
(iii) Between the four half – line paths described in (i) and (ii) there are four domains R_1, R_3, R_3, R_4, each containing an infinite family of semi – hyperbolic like paths which do not approach (0, 0) as $t \to +\infty$ or as $t \to -\infty$, but which become asymptotic to one or another of the four half – line paths as $t \to \infty$ and as $t \to -\infty$.

3.4.3.. Focal point/Spiral point. The isolated critical point (0, 0) is called a spiral point (or focal point) if there exists a neighbourhood of (0, 0) such that every path *C* in this neighbourhood has the following properties.

(i) *C* is defined for all $t > t_0$ (or for all $t < t_0$) for some number t_0.

(ii) *C* approaches (0, 0) as $t \to +\infty$ (or as $t \to -\infty$).

(iii) *C* approaches (0, 0) in a spiral – like manner, winding around (0, 0) an infinite number of times as $t \to +\infty$ (or as $t \to -\infty$).



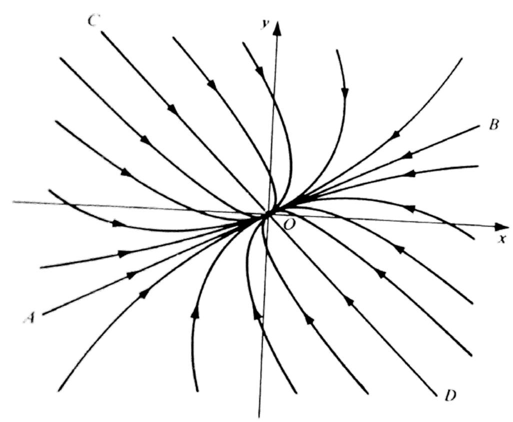
The critical point (0, 0) in the above figure is a spiral point (or focal point). This point is approached in a spiral – like manner by an infinite family of paths as $t \to \infty$ (or as $t \to -\infty$). Here, while the paths approach (0, 0), they do not enter it. That is a point R tracing such a path C approaches O (0, 0) as $t \to \infty$ (or as $t \to -\infty$), but the line or does not tend to a definite direction, since the path constantly winds about O.

3.4.4. Node. The isolated critical point (0, 0) is called a node if there exist a neighbourhood of (0, 0) such that every path C in this neighbourhood has the following properties.

(i) C is defined for all $t > t_0$ (or for all $t < t_0$) for some number t_0.

(ii) C approaches (0, 0) as $t \to +\infty$ (or as $t \to -\infty$).

(iii) C enters (0, 0) as $t \to +\infty$ [or as $t \to -\infty$].



The critical point (0, 0) in the above figure is a node. This point is not only approached but also entered by an infinite family of paths as $t \to \infty$ (or as $t \to -\infty$). That is, a point R tracing such a path not only approaches O but does so in such a way that the line OR tends to a definite direction as $t \to +\infty$ (or as $t \to -\infty$). In above figure, there are four rectilinear paths AO, BO, CO and DO. All other paths are like "semiparabolas" As each of these semiparabolic – like paths approaches O, its slope approaches that of the line AB.

3.5. Stability. Let (0, 0) is an isolated critical point of the autonomous system

$$\frac{dx}{dt} = F(x, y), \frac{dy}{dt} = G(x, y) \qquad (1)$$

Let C be a path of (1) and let $x = x(t)$, $y = y(t)$ be a solution of (1) defining C parametrically. Let

$$D(t) = \sqrt{[x(t)]^2 + [y(t)]^2} \qquad (2)$$

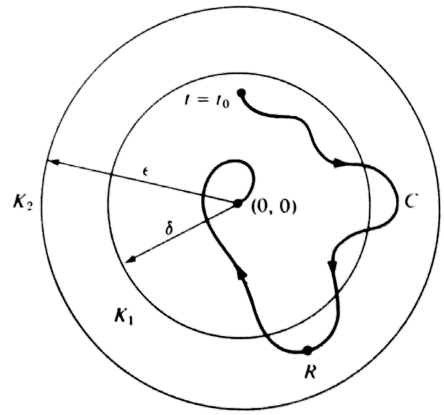
denote the distance between the critical point (0, 0) and the point R. $(x(t), y(t))$ on C. The critical point (0, 0) is called stable if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that the following is true. Every path C for which

$$D(t_0) < \delta \text{ for some } t_0 \qquad (3)$$

is defined for all $t \geq t_0$ and is such that

$$D(t) < \varepsilon \text{ for } t_0 \leq t < \infty \qquad (4)$$

3.5.1. Analysis of the definition. According to (2), the inequality $D(t_0) < \delta$ for some value t_0 in (3), means that the distance between the critical point (0, 0) and the point R on the path C must be less than δ at $t = t_0$. This means that at $t = t_0$, R lies within the circle K_1 of radius δ about (0, 0). Similarly the inequality $D(t) < \varepsilon$ for $t_0 \leq t < \infty$ in (4) means that the distance between (0, 0) and R is less than ε for all $t \geq t_0$, and hence that for $t \geq t_0$, R lies within the circle K_2 of radius ε about (0, 0). Now if (0, 0) is stable, then every path C which is inside the circle K_1 of radius δ at $t = t_0$ will remain inside the circle K_2 of radius $t \geq t_0$.



3.5.2. Asymptotic Stability. Let (0, 0) is an isolated critical point of the system

$$\frac{dx}{dt} = F(x, y), \ \frac{dy}{dt} = G(x, y) \qquad (1)$$

Let C be a path of (1) and let $x = x(t)$, $y = y(t)$ be a solution of (1) representing C parametrically. Let

$$D(t) = \sqrt{[x(t)]^2 + [y(t)]^2} \qquad (2)$$

denote the distance between the critical point (0, 0) and the point R. $(x(t), y(t))$ on C. The critical point (0, 0) is called asymptotically stable if

(i) It is stable and

(ii) There exist a number $\delta_0 > 0$ such that if

$$D(t_0) < \delta_0 \qquad (3)$$

for some value t_0, then

$$\lim_{t\to\infty} x(t) = 0, \ \lim_{t\to\infty} y(t) = 0 \qquad (4)$$

3.5.3. Unstable Critical Point. A critical point is called unstable if it not stable.

Remark. According to definitions, centre, spiral point and node are stable. Out of these three, the spiral point and the node are asymptotically stable. If the directions of the paths in the figures of spiral point and node are reversed, then they become unstable. Saddle point is unstable.

3.5.4. Critical points and paths of linear systems. We consider the linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}\right\} \qquad (1)$$

where *a*, *b*, *c* . *d* are real constants. We attempt to determine a solution of the form

$$\left.\begin{aligned} x &= A\ e^{\lambda t} \\ y &= B\ e^{\lambda t} \end{aligned}\right\} \qquad (2)$$

where *A*, *B* and λ are constants.

If we put (2) in (1), then

$$A\lambda\, e^{\lambda t} = aA\, e^{\lambda t} + bB\, e^{\lambda t}$$

$$B\lambda\, e^{\lambda t} = cA\, e^{\lambda t} + dB\, e^{\lambda t}$$

which gives

$$\left.\begin{aligned} (a-\lambda)A + bB &= 0 \\ a\,A + (b-\lambda)\,B &= 0 \end{aligned}\right\} \qquad (3)$$

This system obviously has the trivial solution $A = B = 0$. But this would give only the trivial solution $x = 0$, $y = 0$ of the system (1). Thus we seek non – trivial solution of the system (3). A necessary and sufficient condition that this system have a non-trivial solution is that the determinant

$$\begin{vmatrix} a-\lambda & b \\ a & d-\lambda \end{vmatrix} = 0 \qquad (4)$$

This gives the quadratic equation

$$\lambda^2 - (a+d)\,\lambda + (ad - bc) = 0 \qquad (5)$$

This equation is called the characteristic equation associated with the system (1). Its roots λ_1 and λ_2 are called the characteristic roots.

If the pair (2) is to be a solution of the system (1), then λ in (2) must be one of these roots.

Result. If λ_1 and λ_2 are roots of characteristic equation (5) then the general solution of the system (1) may be written as

$$x = c_1\, A_1\, e^{\lambda_1 t} + c_2\, A_2\, e^{\lambda_2 t}$$

$$y = c_1\, B_1\, e^{\lambda_1 t} + c_2\, B_2\, e^{\lambda_2 t}$$

where A_1, B_1, A_2 and B_2 are definite constants and c_1, c_2 arbitrary constants.

Remark. Let λ_1 and λ_2 be the roots of the characteristic equation (5). We shall prove that the nature of the critical point (0, 0) of the system (1) depends upon the nature of roots λ_1 and λ_2.

3.5.5. Theorem. Consider the linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + \text{by} \\ \frac{dy}{dt} &= cx + dy \end{aligned}\right\} \qquad (1)$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$, so that (0, 0) is the only critical point of the system. The roots λ_1 and λ_2 of the characteristic equation $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$ are real, unequal and of the same sign. Then the critical point (0, 0) of the linear system (1) is a node.

Proof. We first assume that λ_1 and λ_2 are both negative and take $\lambda_1 < \lambda_2 < 0$. We know that the general solution of the system (1) may then be written

$$\left.\begin{aligned} x &= c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t} \\ y &= c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t} \end{aligned}\right\} \qquad (2)$$

where A_1, B_1, A_2 and B_2 are definite constants and $A_1 B_2 \neq A_2 B_1$ and where c_1 and c_2 are arbitrary constants.

Choosing $c_2 = 0$ we obtain the solutions

$$\left.\begin{aligned} x &= c_1 A_1 e^{\lambda_1 t} \\ y &= c_1 B_1 e^{\lambda_1 t} \end{aligned}\right\} \qquad (3)$$

Choosing $c_1 = 0$, we obtain the solutions

$$\left.\begin{aligned} x &= c_2 A_2 e^{\lambda_2 t} \\ y &= c_2 B_2 e^{\lambda_2 t} \end{aligned}\right\} \qquad (4)$$

For any $c_1 > 0$, the solutions (3) represent a path consisting of "half" of the line $B_1 x = A_1 y$ of slope B_1/A_1. For any $c_1 < 0$, they represent a path consisting of the "other half" of this line. Since $\lambda_1 < 0$, both of these half – line paths approach (0, 0) as $t \to \infty$. Also since $y/x = B_1/A_1$, these two paths enter (0, 0) with slope B_1/A_1.

Similarly, for any $c_2 > 0$ the solutions (4) represent a path consisting of half of the line $B_2 x = A_2 y$; while for any $c_2 < 0$, the path so represented consists of the other half of this line. These two half-line paths also approach (0, 0) as $t \to +\infty$ and enter it with slope B_2/A_2.

Thus the solutions (3) and (4) provide us with four half – line paths which all approach and enter (0, 0) as $t \to +\infty$.

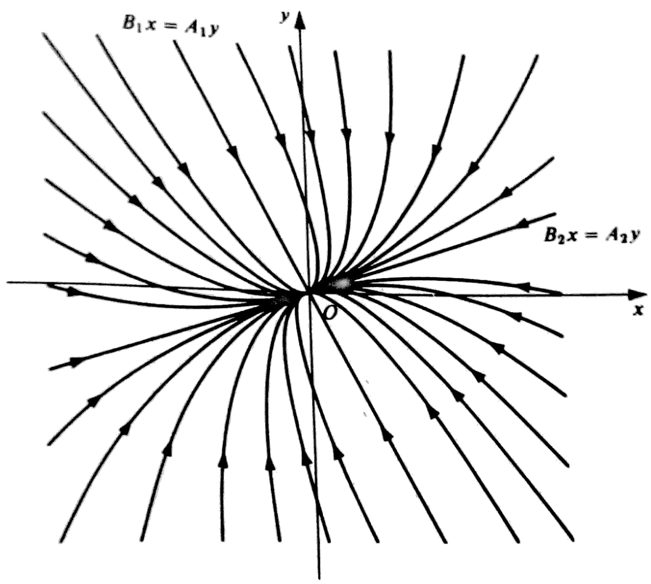
If $C_1 \neq 0$ and $C_2 \neq 0$ the general solution (2), represents non – rectilinear paths.

Again, since $\lambda_1 < \lambda_2 < 0$,

all of these paths approach (0, 0) as $t \to +\infty$. Further

$$\frac{y}{x} = \frac{c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t}}{c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_1 t}} = \frac{\left(c_1 B_1/c_2\right) e^{(\lambda_1 - \lambda_2)t} + B_2}{\left(c_1 A_1/c_2\right) e^{(\lambda_1 - \lambda_2)t} + A_2}$$

Therefore, $\lim_{t \to \infty} \frac{y}{x} = \frac{B_2}{A_2}$ and so all of these paths enter (0, 0) with limiting slope B_2/A_2. Thus all the paths rectilinear as well as non – rectilinear enter (0, 0) as $t \to +\infty$, and all except the two rectilinear paths defined by (3) enter with slope B_2/A_2. According to the definition the critical point (0, 0) is a node. Clearly it is asymptotically stable. A qualitative diagram is given in the below figure.



If now λ_1 and λ_2 are both positive and we take $\lambda_1 > \lambda_2 > 0$, we see that the general solution of (1) is still of the form (2) and particular solutions of the forms (3) and (4) still exist. The case is the same as before, except all the paths approach and enter (0, 0) as $t \to -\infty$. The qualitative diagram of above figure is unchanged except that all the arrows now point in the opposite directions. The critical point (0, 0) is still a node, but in this case, it is clear that it is unstable.

3.5.6. Theorem. Consider the linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}\right\} \qquad (1)$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, so that (0, 0) is the only critical point of the system. The roots λ_1 and λ_2 of the characteristic equation $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$ are real, unequal and of opposite sign. Then the critical point (0, 0) of the linear system (1) is a saddle point.

Proof. We assume that $\lambda_1 < 0 < \lambda_2$. The general solution of (1) may be written as

$$\left.\begin{aligned} x &= c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t} \\ y &= c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t} \end{aligned}\right\} \qquad (2)$$

where A_1, B_1, A_2 and B_2 are definite constants and $A_1 B_2 \neq A_2 B_1$ and where c_1 and c_2 are arbitrary constants.

Choosing $c_2 = 0$, we obtain the solutions

$$\left.\begin{aligned} x &= c_1 A_1 e^{\lambda_1 t} \\ y &= c_1 B_1 e^{\lambda_1 t} \end{aligned}\right\} \qquad (3)$$

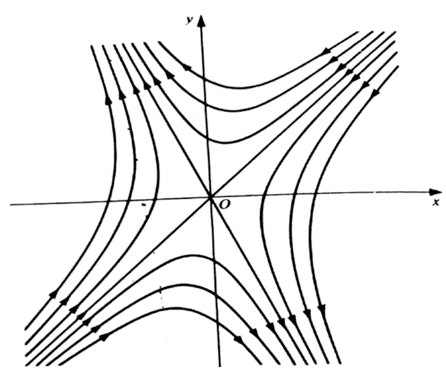
Choosing $c_1 = 0$, we obtain the solutions

$$\left.\begin{aligned} x &= c_2 A_2 e^{\lambda_2 t} \\ y &= c_2 B_2 e^{\lambda_2 t} \end{aligned}\right\} \qquad (4)$$

For any $c_1 > 0$, the solutions (3) represent a path consisting of half of the line $B_1x = A_1y$ of slope B_1/A_1. For any $c_1 < 0$, they represent a path consisting of the other half of this line. Since $\lambda_1 < 0$, both of these half – line paths approach (0, 0) as $t \to \infty$. Similarly, for any $c_2 > 0$ the solutions (4) represent a path consisting of half of line $B_2x = A_2y$, while for any $c_2 < 0$, the path so represented consists of the other half of this line. But in this case, since $\lambda_2 > 0$, both of these half – line paths now approach and enter (0, 0) as $t \to -\infty$.

If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (2) represents non – rectilinear paths. Here since $\lambda_1 < 0 < \lambda_2$, none of these paths can approach (0, 0) as $t \to +\infty$ or as $t \to -\infty$. Further, none of them pass through (0, 0) for any t_0 such that $-\infty < t_0 < \infty$. As $t \to \infty$, we see from (2) that each of these non–rectilinear paths becomes asymptotic to one of the half – line paths defined by (4). As $t \to -\infty$, each of them becomes asymptotic to one of the paths defined by (3).

Thus there are two half – line paths which approach and enter (0, 0) as $t \to \infty$ and two other half – line paths which approach and enter (0, 0) as $t \to -\infty$. All other paths are non – rectilinear paths which do not approach (0, 0) as $t \to +\infty$ or as $t \to -\infty$ but which become asymptotic to one or another of the four half – line paths as $t \to +\infty$, and as $t \to -\infty$. Hence, the critical point (0, 0) is saddle point according to the definition. Clearly, it is unstable. A qualitative diagram is given below,



3.5.7. Theorem. Consider the linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}\right\} \qquad (1)$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, so that (0, 0) is the only critical point of the system. The roots λ_1 and λ_2 of the characteristic equation

$$\lambda^2 - (a + d)\,\lambda + (ad - bc) = 0 \qquad (2)$$

are real and equal. Then the critical point (0, 0) of the linear system (1) is a node.

Proof. We first assume that $\lambda_1 = \lambda_2 = \lambda < 0$. We consider two sub cases:

(i) $a = d \neq 0$ and $b = c = 0$

(ii) All other possibilities which gives a double root. We consider first the sub case – (i).

The characteristic equation (2) becomes

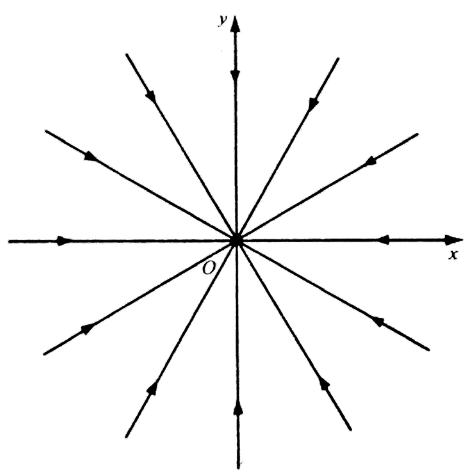
$$\lambda^2 - 2a\lambda + a^2 = 0$$

and hence $\lambda = a = d$. The system (1) is now $\frac{dx}{dt} = \lambda x$, $\frac{dy}{dt} = \lambda y$

The general solution of this system is clearly

$$\left.\begin{aligned} x &= c_1 e^{\lambda t} \\ y &= c_2 e^{\lambda t} \end{aligned}\right\} \qquad (3)$$

where c_1 and c_2 are arbitrary constants. The paths defined by (3) for the various values of c_1 and c_2 are half – lines of all possible slopes. Since $\lambda < 0$, we see that each of these half – lines approaches and enters (0, 0) as $t \to +\infty$. That is, all the paths of the system enter (0, 0) as $t \to +\infty$. Hence the critical point (0, 0) is a node by definition. Clearly, it is asymptotically stable. A qualitative diagram of the path appears in the figure below.



If $\lambda > 0$, the situation is same except that the path enter (0, 0) as $t \to -\infty$, the node (0, 0) is unstable and the arrows in the above figure are all reversed.

This type of node is sometimes called star – shaped node.

Now we consider the sub case (ii). Here the characteristic equation has the double root $\lambda < 0$, but excluding the special circumstances of (i).

Then, we know that the general solution of system (1) may be written as

$$\left.\begin{aligned} x &= c_1 A e^{\lambda t} + c_2 (A_1 t + A_2)\, e^{\lambda t} \\ y &= c_1 B\, e^{\lambda t} + c_2 (B_1 t + B_2)\, e^{\lambda t} \end{aligned}\right\} \qquad (3)$$

where A's and B's are definite constants, c_1 and c_2 are arbitrary constants and $B_1/A_1 = B/A$.

choosing $C_2 = 0$ in (3), we obtain

$$\left.\begin{aligned} x &= c_1 \mathrm{A} e^{\lambda t} \\ y &= c_1 B e^{\lambda t} \end{aligned}\right\} \qquad (4)$$

For any $c_1 > 0$, the solutions (4) represent a path consisting of half of the line $Bx = Ay$ of slope B/A and for any $c_1 < 0$, they represent a path which consist of the other half of this line. Since $\lambda < 0$, both of these halves – line paths approach (0, 0) as $t \to +\infty$. Further since $y/x = B/A$, both paths enter (0, 0) with slope B/A.

If $c_2 \neq 0$, the general solution (3) represents non-rectilinear paths. Since $\lambda < 0$, we see from (3) that

$$\lim_{t\to\infty} x(t) = 0, \ \lim_{t\to\infty} y(t) = 0$$

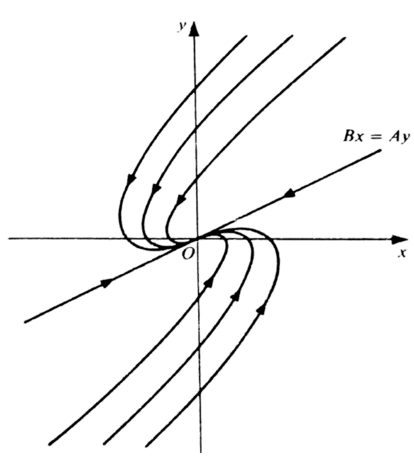
Thus the non – rectilinear paths defined by (3) all approach (0, 0) as $t \to +\infty$. Also

$$\frac{y}{x} = \frac{c_1 B e^{\lambda t} + c_2 (B_1 t + B_2) e^{\lambda t}}{c_1 A e^{\lambda t} + c_2 (A_1 t + A_2) e^{\lambda t}} = \frac{(c_1 B/c_2) + B_2 + B_1 t}{(c_1 A/c_2) + A_2 + A_1 t}$$

So we, have $\lim\limits_{t\to+\infty} \dfrac{y}{x} = \dfrac{B_1}{A_1} = \dfrac{B}{A}$.

Thus all the non – rectilinear paths enter (0, 0) with limiting slope B/A.

Thus all the paths (both rectilinear and non – rectilinear) enter (0, 0) as $t \to +\infty$ with slope B/A. Hence the critical point (0, 0) is a node. Clearly it is asymptotically stable. A qualitative diagram is below.



If $\lambda > 0$, the situation is same except the path enters (0, 0) as $t \to -\infty$, the node (0, 0) is unstable and arrows in this figure are reversed.

3.5.8. Theorem. Consider the linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}\right\} \qquad (1)$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, so that (0, 0) is the only critical point of the system. The roots λ_1 and λ_2 of the characteristic equation $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$ are conjugate complex with real part non-zero i.e. not purely imaginary. Then the critical point (0, 0) of the linear system (1) is a spiral point.

Proof. Since λ_1 and λ_2 are conjugate complex with real part not zero, we write these roots $\alpha \pm i\beta$, where α and β are both real and unequal to zero. Then we know that the general solution of the system (1) may be written as

$$\left.\begin{aligned} x &= e^{\alpha t}\left[c_1\left(A_1\cos\beta t - A_2\sin\beta t\right) + c_2\left(A_2\cos\beta t + A_1\sin\beta t\right)\right] \\ y &= e^{\alpha t}\left[c_1\left(B_1\cos\beta t - B_2\sin\beta t\right) + c_2\left(B_2\cos\beta t + B_1\sin\beta t\right)\right] \end{aligned}\right\} \qquad (2)$$

where A_1, A_2, B_1 and B_2 are definite real constants and c_1 and c_2 are arbitrary constants.

Let us first assume that $\alpha < 0$. Then from (2), we get

$$\lim_{t\to\infty} x(t) = 0 \text{ and } \lim_{t\to\infty} y(t) = 0$$

and hence all the paths defined by (2) approach (0, 0) as $t \to \infty$. We may also write (2) as

$$\left.\begin{aligned} x &= e^{\alpha t}\left(c_3\cos\beta t + c_4\sin\beta t\right) \\ y &= e^{\alpha t}\left(c_5\cos\beta t + c_6\sin\beta t\right) \end{aligned}\right\} \qquad (3)$$

where $c_3 = c_1A_1 + c_2A_2,\ c_4 = c_2A_1 - c_1A_2$

$c_5 = c_1B_1 + c_2B_2,\ c_6 = c_2B_1 - c_1B_2$

Assuming c_1 and c_2 are real, these solutions (3) represent all paths in the real xy – plane. These solutions can be put in the form

$$\left.\begin{aligned} x &= K_1 e^{\alpha t}\cos\left(\beta t + \phi_1\right) \\ y &= K_2 e^{\alpha t}\cos\left(\beta t + \phi_2\right) \end{aligned}\right\} \qquad (4)$$

where $K_1 = \sqrt{c_3^2 + c_4^2}$, $K_2 = \sqrt{c_5^2 + c_6^2}$ and ϕ_1, ϕ_2 are defined by the equations

$$\cos\phi_1 = \frac{c_3}{K_1} \qquad \cos\phi_2 = \frac{c_5}{K_2}$$

$$\sin\phi_1 = -\frac{c_4}{K_1} \quad \sin\phi_2 = -\frac{c_6}{K_2}$$

Now we consider

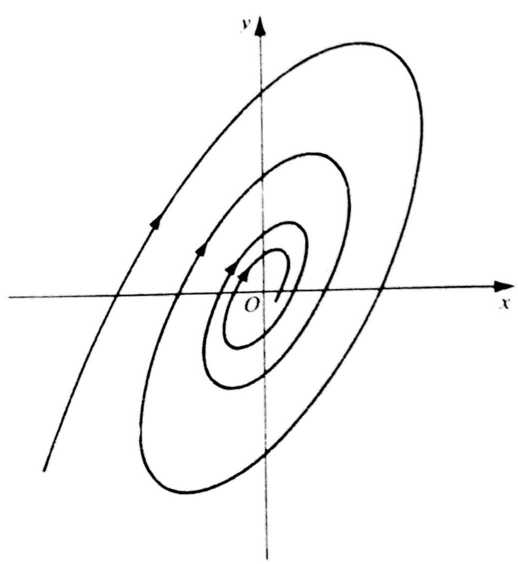
$$\frac{y}{x} = \frac{K_2 e^{\alpha t} \cos(\beta t + \phi_2)}{K_1 e^{\alpha t} \cos(\beta t + \phi_1)} \tag{5}$$

Letting K = K_2/K_1 and $\phi_3 = \phi_1 - \phi_2$, this becomes

$$\frac{y}{x} = \frac{K\cos(\beta t + \phi_1 - \phi_3)}{\cos(\beta t + \phi_1)} = \left[\frac{\cos(\beta t + \phi_1)\cos\phi_3 + \sin(\beta t + \phi_1)\sin\phi_3}{\cos(\beta t + \phi_1)}\right] \tag{6}$$

$$= K[\cos\phi_3 + \sin\phi_3 \tan(\beta t + \phi_1)] \text{ provided } \cos(\beta t + \phi_1) \neq 0.$$

Since trigonometric functions involved in (5) and (6) are periodic (as sine and cosine functions are periodic), so it is clear that $\lim\limits_{t\to\infty} \frac{y}{x}$ does not exist and so the paths do not enter (0, 0). But from (5) and (6), it is clear that the paths approach (0, 0) in a spiral – like manner, winding around (0, 0) an infinite number of times as $t \to +\infty$. So the critical point (0, 0) is a spiral point. Clearly, it is asympototically stable. A qualitative diagram is given below



If $\alpha > 0$, the situation is the same except that the paths approach (0, 0) as $t \to -\infty$, the spiral point (0, 0) is unstable, and the arrows in this figure are reversed.

3.5.9. Theorem. Consider the linear system

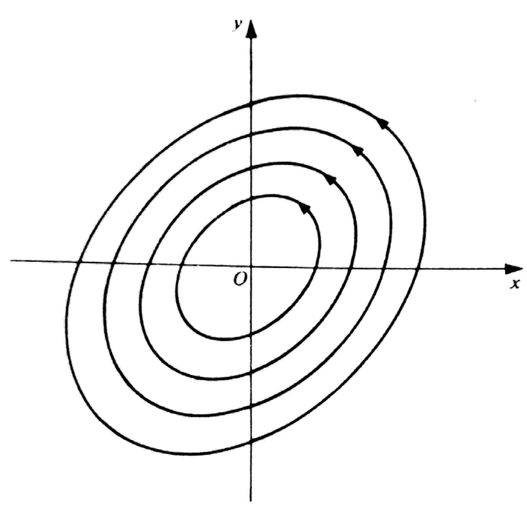
$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}\right] \qquad (1)$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, so that (0, 0) is the only critical point of the system. The roots λ_1 and λ_2 of the characteristic equation $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$ are purely imaginary, then the critical point (0, 0) of the linear system (1) is center.

Proof. Since λ_1 and λ_2 are purely imaginary so we may write them as $\alpha \pm i\beta$, where $\alpha = 0$ but β is real and unequal to zero. Then the general real solution of (1) may be written as

$$\left.\begin{aligned} x &= K_1 \cos(\beta t + \phi_1) \\ y &= K_2 \cos(\beta t + \phi_2) \end{aligned}\right\} \qquad (2)$$

where K_1, K_2, ϕ_1 and ϕ_2 are defined in the theorem – 4. The solutions (2) define the paths in the real xy – phase plane. We know that the trigonometric functions in (2) oscillate indefinitely between + 1 and – 1 as $t \to \infty$ and as $t \to -\infty$, so the paths do not approach (0, 0) as $t \to +\infty$ or as $t \to -\infty$. Rather it is clear from (2) that x and y are periodic functions of t and hence the paths are closed curves surrounding (0, 0), members of which are arbitrarily close to (0, 0). In fact they are infinite family of ellipse with center at (0, 0). Hence the critical point (0, 0) is center. Clearly it is stable. However, since the paths do not approach(0, 0), the critical point is not asymptotically stable. A qualitative diagram of the paths appear in the adjoining figure



3.5.10. Example. Determine the nature of the critical point (0, 0) of the system

$$\left.\begin{aligned}\frac{dx}{dt} &= 2x - 7y \\ \frac{dy}{dt} &= 3x - 8y\end{aligned}\right] \qquad (1)$$

and determine whether or not the point is stable.

Solution. Comparing system (1) with standard system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

we get, $a = 2$, $b = -7$, $c = 3$ and $d = -8$.

We know that characteristic equation is $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$,

that is, $\lambda^2 + 6\lambda + 5 = 0$ (2)

Hence the roots of characteristic equation are $\lambda_1 = -5$ and $\lambda_2 = -1$. Since the roots are real, unequal and of the same sign (both negative), we conclude that the critical point (0, 0) of (1) is a node. Since the roots are real and negative, the point is asymptotically stable.

3.5.11. Example. Consider the linear autonomous system

$$\left.\begin{aligned}\frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= 3x - y\end{aligned}\right\} \qquad (1)$$

(i) Determine the nature of the critical point (0, 0) of this system.

(ii) Find the general solution of this system.

(iii) Find the differential equation of the paths in the *xy*-plane and obtain the general solution of this differential equation.

Solution. (i) Comparing (1) with the system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

Here $a = 1$, $b = 1$, $c = 3$, $d = -1$

The characteristic equation is

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \Rightarrow \lambda^2 - 4 = 0$$

Roots are $\lambda_1 = 2,\ \lambda_2 = -2$.

Since the roots are real, unequal and of opposite sign so the critical point (0, 0) is a saddle point.

(ii) We assume a solution of the form

$$\left.\begin{aligned} x &= Ae^{\lambda t} \\ y &= Be^{\lambda t} \end{aligned}\right\} \qquad (2)$$

Substituting (2) into (1), we obtain

$$\left.\begin{aligned} A\lambda e^{\lambda t} &= Ae^{\lambda t} + Be^{\lambda t} \\ B\lambda e^{\lambda t} &= 3Ae^{\lambda t} - Be^{\lambda t} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} (\lambda-1)\,A - B &= 0 \\ -3A + (\lambda+1)\,B &= 0 \end{aligned}\right\} \qquad (3)$$

For non–trivial solution of this system, we must have

$$\begin{vmatrix} \lambda-1 & -1 \\ -3 & \lambda+1 \end{vmatrix} = 0 \Rightarrow \lambda_1 = 2,\ \lambda_2 = -2$$

Setting $\lambda = \lambda_1 = 2$ in (3), we obtain

$$A - B = 0 \quad \Rightarrow \quad A - B = 0$$

$$-3A + 3B = 0 \Rightarrow \quad A - B = 0$$

A simple non – trivial solution of this system is obviously $A = B = 1$.

With these values of A, B and λ we find the non-trivial solution

$$x = e^{2t},\ y = e^{2t} \qquad (4)$$

Again, setting $\lambda = \lambda_2 = -2$ in (3), we obtain

$$-3A - B = 0 \Rightarrow 3A + B = 0$$

$$-3A - B = 0 \Rightarrow 3A + B = 0$$

A simple non – trivial solutions of this system is obviously $A = 1$, $B = -3$

With these values of A, B and λ, we find the non – trivial solution

$$x = e^{-2t},\ y = -3e^{-2t} \qquad (5)$$

Using solutions (4) and (5), the general solution may be written as

$$x = c_1 e^{2t} + c_2 e^{-2t},\ y = c_1 e^{2t} - 3c_2 e^{-2t}$$

where c_1 and c_2 are arbitrary constants.

(iii) Eliminating dt from equations (1), we obtain

$$\frac{dy}{dx} = \frac{3x - y}{x + y} \qquad (6)$$

which is a homogeneous first order differential equation,

Put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Hence (6) can be written as $v + x\dfrac{dv}{dx} = \dfrac{3-v}{1+v} \Rightarrow x\dfrac{dv}{dx} = \dfrac{3-2v-v^2}{1+v}$

$$\Rightarrow \quad \frac{1}{2}\frac{2(1+v)}{v^2+2v-3}\,dv = -\frac{dx}{x}$$

Integrating both sides $\dfrac{1}{2}\log(v^2+2v-3) = -\log x + \log c'$

$$\Rightarrow \quad \log(v^2+2v-3) = \log\left(\frac{c}{x^2}\right), \text{ where } c = (c')^2$$

$$\Rightarrow v^2+2v-3 = \frac{c}{x^2} \Rightarrow y^2 + 2xy - 3x^2 = c \text{ [since } y = vx] \qquad (7)$$

where c is an arbitrary constant. Equation (7) is the equation of the family of paths in the xy-phase plane.

3.5.12. Exercises. Determine the nature of the critical point (0, 0) of each of the linear autonomous systems in following exercises. Also, determine whether or not the critical point is stable.

1. $\dfrac{dx}{dt} = 3x + 4y \quad , \quad \dfrac{dy}{dt} = 3x + 2y$ (Saddle point, unstable)

2. $\dfrac{dx}{dt} = x - y \quad , \quad \dfrac{dy}{dt} = x + 5y$ (Node, unstable)

3. $\dfrac{dx}{dt} = x + 3y \quad , \quad \dfrac{dy}{dt} = 3x + y$ (Saddle point, unstable)

4. $\dfrac{dx}{dt} = 2x - 4y \quad , \quad \dfrac{dy}{dt} = 2x - 2y$ (Center, stable)

3.6. Critical points and paths of nonlinear systems. We now consider the nonlinear real autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y) \end{aligned}\right] \qquad (1)$$

We assume that the system (1) has an isolated critical point which we shall choose to be the origin (0, 0). We now assume further that the functions P and Q on the right members of (1) are such that $P(x, y)$ and $Q(x, y)$ can be written in the form

$$\left.\begin{aligned} P(x, y) &= ax + by + P_1(x, y) \\ Q(x, y) &= cx + dy + Q_1(x, y) \end{aligned}\right\} \qquad (2)$$

where

(i) a, b, c and d are real constants and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

(ii) P_1 and Q_1 have continuous first partial derivatives for all (x, y) and are such that

$$\lim_{(x\,,\,y)\to(0\,,\,0)} \frac{P_1(x\,,\,y)}{\sqrt{x^2+y^2}} = 0 \quad \lim_{(x\,,\,y)\to(0\,,\,0)} \frac{Q_1(x\,,\,y)}{\sqrt{x^2+y^2}} = 0 \qquad (3)$$

Thus the system under consideration may be written in the form

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by + P_1(x\,,\,y) \\ \frac{dy}{dt} &= cx + dy + Q_1(x\,,\,y) \end{aligned}\right\} \qquad (4)$$

where a, b, c, d, P_1 and Q_1 satisfy the requirements (i) and (ii) above.

Also, if $P(x, y)$ and $Q(x,y)$ in (1) can be expanded in power series about (0, 0), the system (1) takes the form

$$\left.\begin{aligned} \frac{dx}{dt} &= \left[\frac{\partial P}{\partial x}\right]_{(0\,,\,0)} x + \left[\frac{\partial P}{\partial y}\right]_{(0\,,\,0)} y + a_{12}x^2 + a_{22}xy + a_{21}y^2 + \ldots \\ \frac{dy}{dt} &= \left[\frac{\partial Q}{\partial x}\right]_{(0\,,\,0)} x + \left[\frac{\partial Q}{\partial y}\right]_{(0\,,\,0)} y + b_{12}x^2 + b_{22}xy + b_{21}y^2 + \ldots \end{aligned}\right\} \qquad (5)$$

This system is of the form (4), where $P_1(x, y)$ and $Q_1(x, y)$ are the terms of higher degree in the rigt members of the equations. The requirements (i) and (ii) will be satisfied provided the

Jacobian $\left[\dfrac{\partial(P\,,\,Q)}{\partial(x\,,\,y)}\right]_{(0\,,\,0)} \neq 0$.

Note that the constant term are missing in the expansions in the right members of (5) because

$P(0, 0) = Q(0, 0) = 0$.

3.6.1. Example. The system $\dfrac{dx}{dt} = x + 2y + x^2$, $\dfrac{dy}{dt} = -3x - 4y + 2y^2$ is of the form (4) and satisfies the requirements (i) and (ii) above. Here $a = 1$, $b = 2$, $c = -3$, $d = -4$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 2 \neq 0$.

Further, $P_1(x, y) = x^2$, $Q_1(x, y) = 2y^2$ and hence

$$\lim_{(x\,,\,y)\to(0\,,\,0)} \frac{P_1(x\,,\,y)}{\sqrt{x^2+y^2}} = \lim_{(x\,,\,y)\to(0\,,\,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$$

and $\lim_{(x, y)\to(0, 0)} \frac{Q_1(x, y)}{\sqrt{x^2+y^2}} = \lim_{(x, y)\to(0, 0)} \frac{2y^2}{\sqrt{x^2+y^2}} = 0$

By the requirement (ii) of the non – linear term $P_1(x, y)$ and $Q_1(x, y)$ in (4) tend to zero more rapidly than the linear terms ax + by and $cx + dy$. Hence one would suspect that the behaviour of the paths of the system (4) near (0, 0) would be similar to that of the paths of the related linear system

$$\frac{dx}{dt} = ax + by \quad , \quad \frac{dy}{dt} = cx + dy$$

obtained from (4) by neglecting the non linear terms.

3.6.2. Theorem (Without Proof).

Hypothesis. Consider the linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by + P_1(x, y) \\ \frac{dy}{dt} &= cx + dy + Q_1(x, y) \end{aligned}\right\} \qquad (1)$$

where a, b, c, d, P_1 and Q_1 satisfy the requirements (i) and (ii) above. Consider also the corresponding linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}\right\} \qquad (2)$$

obtained from (1) by neglecting the non – linear terms $P_1(x, y)$ and $Q_1(x, y)$. Both systems have an isolated critical point at (0, 0). Let λ_1 and λ_2 be the roots of the characteristic equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \qquad (3)$$

of the linear system (2).

Conclusions. (1) The critical point (0, 0) of the non – linear system (1) is of the same type as that of the linear system (2) in the following cases.-

(i) If λ_1 and λ_2 are real, unequal and of the same sign, then not only (0, 0) is a node of (2), but also (0, 0) is a node of (1).

(ii) If λ_1 and λ_2 are real, unequal and of opposite sign, then not only (0, 0) is a saddle point of (2), but also (0, 0) is a saddle point of (1).

(iii) If λ_1 and λ_2 are real and equal and the system (2) is not such that $a = d \neq 0$, $b = c = 0$, then not only (0, 0) is a node of (2) but also (0, 0) is a node of (1).

(iv) If λ_1 and λ_2 are conjugate complex with real part not zero, then not only (0, 0) is a spiral point of (2), but also (0, 0) is a spiral point of (1).

(II) The critical point (0, 0) of the non – linear system (1) is not necessarily of the same type as that of the linear system (2) in the following cases.-

(v) If λ_1 and λ_2 are real and equal and the system (2) is such that $a = d \neq 0$, $b = c = 0$, then although (0, 0) is a node of (2), the point (0, 0) may be either a node or a spiral point of (1).

(vi) If λ_1 and λ_2 are pure imaginary, then although (0, 0) is a center of (2), the point (0, 0) may be either a center or a spiral point of (1).

3.6.3. Theorem (Without Proof). Assuming the hypothesis of the above theorem we can find following conclusions concerning the stability of the critical point.

(i) If both roots of the characteristic equation of the linear system (2) are real and negative or conjugate complex with negative real parts, then not only (0, 0) an asymptotically stable critical point of (2) but also (0, 0) is an asymptotically stable critical point of (1).

(ii) If the roots of the characteristic equation are pure imaginary, then although (0, 0) is a stable critical point of (2), it is not necessarily a stable critical point of (1). Indeed the critical point (0, 0) of (1) may be asymptotically stable, but not asymptotically stable, or unstable.

(iii) If either of the roots of characteristic equation is real and positive or is the roots are conjugate complex with positive real parts, then not only (0, 0) an unstable critical point of (2), but also (0, 0) is an unstable critical point of (1).

3.6.4. Example. Consider the non – linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= x + 4y - x^2 \\ \frac{dy}{dt} &= 6x - y + 2xy \end{aligned}\right\} \qquad (1)$$

This is of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by + P_1(x, y) \\ \frac{dy}{dt} &= cx + dy + Q_1(x, y) \end{aligned}$$

where $a = 1$, $b = 4$, $c = 6$, $d = -1$

$P_1(x, y) = -x^2$, $Q_1(x, y) = 2xy$.

Here requirements (i) and (ii) of above theorems i.e

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \text{ and } \lim_{(x, y)\to(0, 0)} \frac{P_1(x, y)}{\sqrt{x^2+y^2}} = \lim_{(x, y)\to(0, 0)} \frac{Q_1(x, y)}{\sqrt{x^2+y^2}} = 0$$

are both satisfied as

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 6 & -1 \end{vmatrix} = -25 \neq 0$ and $\lim\limits_{(x,y)\to(0,0)} \dfrac{-x^2}{\sqrt{x^2+y^2}} = \lim\limits_{x\to 0} \dfrac{-x^2}{\sqrt{x^2+m^2x^2}} = 0.$

Also, $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{2mx^2}{\sqrt{x^2+m^2x^2}} = 0$

Hence to investigate the critical point (0, 0) of (1), we consider the linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= x + 4y \\ \frac{dy}{dt} &= 6x - y \end{aligned}\right\} \qquad (2)$$

The characteristic equation of (2) is

$$\lambda^2 - 25 = 0 \Rightarrow \lambda = 5, -5.$$

Since the roots are real, unequal and of opposite sign, so the critical point (0, 0) is a saddle point of (2) and by above theorem, it is also a saddle point of non – linear system (1).

Also, clearly this point is unstable.

Eliminating dt from equations (1), we obtain the differential equation

$$\frac{dy}{dx} = \frac{6x - y + 2xy}{x + 4y - x^2} \qquad (3)$$

which gives the slope of the paths in *xy* phase plane defined by the solutions of (1). The differential equation (3), can be written as

$$(6x - y + 2xy)\,dx - (x + 4y - x^2)\,dy = 0$$

Comparing it with $Mdx + Ndy = 0$

$$M = 6x - y + 2xy \qquad N = -(x + 4y - x^2)$$

Then $\dfrac{\partial M}{\partial y} = -1 + 2x$, $\dfrac{\partial N}{\partial x} = -1 + 2x$

Therefore, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. Hence differential equation (3) is an exact differential equation.

Its solution is $\displaystyle\int_{y\,=\,\text{constant}} Mdx + \int(\text{Terms in } N \text{ not containing } x)\,dy = C$

$$\Rightarrow \int(6x - y + 2xy)\,dx + \int -4y\,dy = C$$

$$\Rightarrow \quad 3x^2 - xy + x^2y - 2y^2 = C \qquad (4)$$

where c is an an arbitrary constant. Equation (4) is the equation of the family of paths in the *xy* phase plane.

3.7. Check Your Progress.

Determine the nature of the critical point (0, 0) of the system

1. $$\left.\begin{aligned}\frac{dx}{dt} &= 2x + 4y \\ \frac{dy}{dt} &= -2x + 6y\end{aligned}\right]$$

2. $$\left.\begin{aligned}\frac{dx}{dt} &= \sin x - 4y \\ \frac{dy}{dt} &= \sin 2x - 5y\end{aligned}\right\}$$

3.8. Summary.

On behalf of the results obtained in the theorems, the following table can be considered as the concluding one for the critical point of a plane autonomous system

S. No.	Nature of Roots λ_1, λ_2	Nature of Critical point	Stability of critical point
1.	Real, unequal, same sign	Node	Asympototically stable if roots are negative and unstable if roots are positive.
2.	Real,unequal, opposite sign	Saddle point	Unstable.
3.	Real and equal.	Node	Asympototically stable if roots are negative and unstable if roots are positive.
4.	Conjugate complex but not pure imaginary.	Spiral point	Asympototically stable if real part of roots is negative and unstable if real part of roots is positive.
5.	Pure imaginary	Center	Stable but not asymptotically stable.

Books Suggested:

1. Ross, S.L., Differential equations, John Wiley and Sons Inc., New York, 1984.
2. Boyce, W.E., Diprima, R.C., Elementary differential equations and boundary value problems, John Wiley and Sons, Inc., New York, 4th edition, 1986.
3. Simmon, G.F., Differential Equations, Tata McGraw Hill, New Delhi, 1993.

4

Liapunov Function, Limit Cycles and Sturm Liouville BVP

Structure

4.1. Introduction. This chapter contains important results to check the stability and asymptotic stability of critical point of an autonomous system. Also the concept of periodic solutions for a linear homogeneous system, limit cycles, Sturm Liouville BVP are discussed.

4.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) The methods to obtain solution of a Sturm Liouville BVP.

(ii) The methods to check the stability and asymptotic stability of critical points.

(iii) The concepts of limit cycles.

4.1.2. Keywords. Liouville Function, SLBVP, Limit Cycles, Periodic Solutions.

4.2. Liapunov's direct method.

Let $E(x, y)$ have continuous first partial derivatives at all points (x, y) in a domain D containing the origin (0, 0).

(1) The function E is called positive definite in D if $E(0, 0) = 0$ and $E(x, y) > 0$ for all other points (x, y) in D.

(2) The function E is called positive semi definite in D if $E(0, 0) = 0$ and $E(x, y) \geq 0$ for all other points (x, y) in D.

(3) The function E is called negative definite in D if $E(0, 0) = 0$ and $E(x, y) < 0$ for all other points (x, y) in D.

(4) The function E is called negative semi definite in D if $E(0, 0) = 0$ and $E(x, y) \leq 0$ for all other points (x, y) in D.

For example,

1. The function E defined by $E(x, y) = x^2 + y^2$ is positive definite in every domain D containing (0, 0). Clearly $E(0, 0) = 0$ and $E(x, y) > 0$ for all $(x, y) \neq (0, 0)$.
2. The function E defined by $E(x, y) = x^2$ is positive semi definite in very domain D containing (0, 0). Note that $E(0, 0) = 0$, $E(0, y) = 0$ for all $(0, y)$ such that $y \neq 0$ in D, and $E(x, y) > 0$ for all (x, y) such that $x \neq 0$ in D. There are no other points in D and so we see that $E(0, 0) = 0$ and $E(x, y) \geq 0$ for all other points in D.
3. In like manner, we see that the function E defined by $E(x, y) = -x^2 - y^2$ is negative definite in D and that defined by $E(x, y) = -x^2$ is negative semi definite in D.

4.2.1. Derivative with Respect to Given System. Consider the non – linear autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y) \end{aligned}\right\} \qquad (1)$$

Assume that this system has an isolated critical point at the origin (0, 0) and that P and Q have continuous first partial derivatives for all (x, y). Let $E(x, y)$ have continuous first partial derivatives at all points (x, y) in a domain D containing the origin (0, 0). The derivative of E with respect to the system (1) is the function $\dot{E}$ defined by

$$\dot{E}(x, y) = \frac{\partial E(x, y)}{\partial x} P(x, y) + \frac{\partial E(x, y)}{\partial y} Q(x, y)$$

For example, consider the system

$$\left.\begin{aligned} \frac{dx}{dt} &= -x + y^2 \\ \frac{dy}{dt} &= -y + x^2 \end{aligned}\right\} \qquad (1)$$

and the function E defined by $E(x, y) = x^2 + y^2$.

Here, we have $P(x, y) = -x + y^2$, $Q(x, y) = -y + x^2$, $E(x, y) = x^2 + y^2$. Then derivative of E w.r.t. the system (1) is given by

$$\dot{E}\ (x, y) = \frac{\partial E}{\partial x} P(x, y) + \frac{\partial E}{\partial y} Q(x, y) = 2x(-x + y^2) + 2y(-y + x^2) = -2x^2 - 2y^2 + 2x^2y + 2xy^2.$$

Remark. Let C be a path of non – linear system (1) and let $x = x(t)$, $y = y(t)$ be an arbitrary solution of (1) defining C parametrically; and let $E(x, y)$ have continuous first partial derivatives for all (x, y) in a domain containing C. Then E is a composite function of t along C, and using the chain rule, we find that the derivative of E w.r.t. to t along C is

$$\frac{dE[x(t), y(t)]}{dt} = E_x\{x(t), y(t)\ \frac{dx(t)}{dt} + E_y\{x(t), y(t)\ \frac{dy(t)}{dt}$$

$$= E_x[x(t), y(t)\ P[x(t), y(t)] + E_y[x(t), y(t)]\ Q[x(t), y(t)]$$

$$= \dot{E}[x(t), y(t)]$$

Thus we see that the derivative of $E\ (x(t), y(t))$ with respect to t along the path C is equal to the derivative of E w.r.t. to the system (1), evaluated at $x = x(t)$ and $y = y(t)$.

4.2.2. Liapunov Function. Consider the system (non – linear)

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y) \end{aligned}\right\} \qquad (1)$$

Assume that this system has an isolated critical point at the origin (0, 0) and that P and Q have continuous first partial derivatives for all (x, y). Let $E(x, y)$ be positive definite for all (x, y) in domain D containing the origin and such that the derivative $\dot{E}(x, y)$ of E with respect to the system (1) is negative semi definite for all $(x, y) \in D$. Then $E(x, y)$ is called a Liapunov function for the system (1) in D.

4.2.3. Example. Consider the system

$$\left.\begin{aligned} \frac{dx}{dt} &= -x + y^2 \\ \frac{dy}{dt} &= -y + x^2 \end{aligned}\right\} \qquad (1)$$

and the function E defined by

$$E(x, y) = x^2 + y^2 \qquad (2)$$

It is clear that $E(x, y)$ is positive definite in every domain D containing (0, 0). In previous example, we have found

$$\dot{E}\ (x, y) = -2(x^2 + y^2) + 2(x^2y + xy^2) \qquad (3)$$

for all (x, y). If this is negative semi – definite for all (x, y) in some domain D containing (0, 0), then E defined by (2) is a Liapunov function for the system (1).

Clearly $\dot{E}(0, 0) = 0$. Now observe the following:

If $x < 1$ and $y \neq 0$, then $xy^2 < y^2$ and if $y < 1$ and $x \neq 0$, then $x^2y < x^2$. Thus if $x < 1$, $y < 1$ and $(x, y) \neq (0, 0)$ then $x^2y + xy^2 < x^2 + y^2$ and hence $-(x^2 + y^2) + (x^2y + xy^2) < 0$

Thus in every domain D containing (0, 0) and such that $x < 1$ and $y < 1$, $\dot{E}(x, y)$ given by (3) is negative definite and hence negative semi definite. Thus E defined by (2) is a Liapunov function for the system (1).

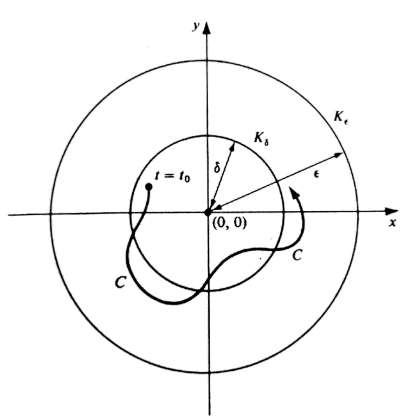
4.2.4. Theorem. Consider the non – linear autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x\ ,\ y) \\ \frac{dy}{dt} &= Q(x\ ,\ y) \end{aligned}\right\} \qquad (1)$$

Assume that this system has an isolated critical point at the origin (0, 0) and that P and Q have continuous first partial derivatives for all (x, y). If there exist a Liapunov function $E(x, y)$ for the system (1) in some domain D containing (0, 0), then the critical point (0, 0) of (1) is stable.

Proof. Let K_ε be a circle of radius $\varepsilon > 0$ with centre at the critical point (0, 0), where $\varepsilon > 0$ is small enough so that this circle K_ε lies entirely in the domain D. From a theorem of real analysis, we know that a real valued function which is continuous on a closed bounded set assumes both a maximum value and a minimum value on that set. Since the circle K_ε is closed bounded set in the plane and E is continuous in D and hence on K_ε, so by above mentioned theorem, E assumes, in particular, a minimum value on K_ε.

Now since $E(x, y)$ is a Liapunov function, So, E is positive definite in D and so minimum value of E must be positive. Thus E assumes a positive minimum m on the circle K_ε. Next observe that since E is continuous at (0, 0) and $E(0, 0) = 0$, there exists a positive number δ satisfying $\delta < \varepsilon$ such that $E(x, y) < m$ for all (x, y) within or on the circle K_δ of radius δ and center at (0, 0).



Now let C be any path of (1) and let $x = x(t)$, $y = y(t)$ be an arbitrary solution of (1) defining C parametrically and suppose C defined by $[x(t), y(t)]$ is at a point within the inner circle K_δ at $t = t_0$. Then

$$E[x(t_0), y(t_0)] < m$$

Now since $\dot{E}$ is negative semi-definite in D (as E is a Liapunov function) and we know that the derivative of $E[x(t), y(t)]$ w.r.t. 't' along the path C is equal to the derivative of E w.r.t. to the system (1) evaluated at

$$x = x(t), y = y(t)$$

that is,
$$\frac{dE[x(t)\ ,\ y(t)]}{dt} = \dot{E}\,[x(t), y(t)]$$

But
$$\dot{E}\,[x(t), y(t)] \leq 0 \qquad \text{[Since } \dot{E} \text{ is negative semi-definite]}$$

So, we have

$$\frac{dE[x(t)\ ,\ y(t)]}{dt} \leq 0 \text{ for } [x(t), y(t)] \in D.$$

Thus $E[x(t), y(t)]$ is a non-increasing function of t along C. Hence

$$E[x(t), y(t)] \leq E[x(t_0), y(t_0)] < m \text{ for all } t > t_0$$

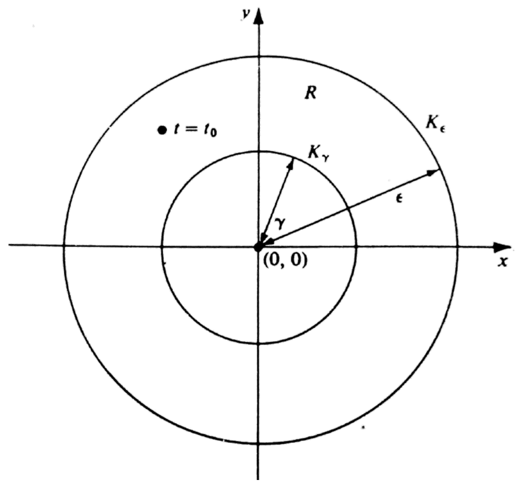
Since, $E[x(t), y(t)]$ would have to be $\geq m$ on the outer circle K_ε, we see that the path C defined by $x = x(t)$, $y = y(t)$ must remain within K_ε for all $t > t_0$. Thus from the definition of stability of the critical point (0, 0), we see that the critical point (0, 0) of (1) is stable.

4.2.5. Theorem. Consider the system

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x\ ,\ y) \\ \frac{dy}{dt} &= Q(x\ ,\ y) \end{aligned}\right\} \qquad (1)$$

Assume that this system has an isolated critical point at the origin (0, 0) and that P and Q have continuous first partial derivatives for all (x, y). If there exists a Liapunov function E for the system (1) in some domain D containing (0, 0) such that E also has the property that $\dot{E}$ is negative definite in D, then the critical point (0, 0) of (1) is asymptotically stable.

Proof. Let K_ε be a circle of radius $\varepsilon > 0$ with center at the critical point (0, 0) and lying entirely in D. Also let C be any path of (1) and let $x = x(t)$, $y = y(t)$ be an arbitrary solution of (1) defining C parametrically and suppose C defined by $[x(t), y(t)]$ is at a point within K_ε at $t = t_0$.



Now since $\dot{E}$ is given negative definite, we have $\dot{E}[x(t), y(t)] < 0$ for $[x(t), y(t)] \in D$

But we know that $\dfrac{dE[x(t)\ ,\ y(t)]}{dt} = \dot{E}\ [x(t), y(t)]$

So, we must have $\dfrac{dE[x(t)\ ,\ y(t)]}{dt} < 0$ for $[x(t), y(t)] \in D$.

Thus, $E[x(t), y(t)]$ is strictly decreasing function of t along C. Since E is positive definite in D, thus $\lim\limits_{t\to\infty} E[x(t), y(t)]$ exists and is some number $L \geq 0$.

We shall prove that $L = 0$.

On the contrary, assume that $L > 0$. Since E is positive definite, there exists a positive number γ satisfying $\gamma < \varepsilon$ such that $E(x, y) < L$ for all (x, y) within the circle K_γ of radius γ and centre (0, 0). By a theorem of real analysis, we know that a real - valued function which is continuous on a closed bounded set assumes both maximum value and a minimum value on that set. We apply this theorem to the continuous function E on the closed region R between and on the two circles K_ε and K_γ. So, $\dot{E}$ must have a maximum value on this region R. But $\dot{E}$ is negative definite in D and hence in this region R which does not include (0, 0), so we see that E assumes a negative maximum $-k$ on R.

Since $E\ [x(t), y(t)]$ is strictly decreasing function of t along C and

$$\lim_{t\to\infty} E[x(t), y(t)] = L$$

so the path C defined by $x = x(t)$, $y = y(t)$ cannot enter the domain K_γ for any $t > t_0$ and so remains in R for all $t \geq t_0$. Thus we have

$$\dot{E}\ [x(t), y(t)] \leq -k \text{ for all } t \geq t_0.$$

But we know that $\dfrac{dE[x(t)\ ,\ y(t)]}{dt} = \dot{E}\ [x(t), y(t)]$

So, we obtain $\dfrac{dE[x(t)\ ,\ y(t)]}{dt} \le -k$ for all $t \ge t_0$ (2)

Now consider the identity

$$E[x(t), y(t)] - E[x(t_0), y(t_0)] = \int_{t_0}^{t} \frac{dE[x(t)\ ,\ y(t)]}{dt}\, dt \qquad (3)$$

Using (2) on *R*.H.S. of (3), we get

$$E[x(t), y(t)] - E[x(t_0), y(t_0)] \le \int_{t_0}^{t} k\, dt$$

$$\Rightarrow \quad E[x(t), y(t)] \le E[x(t_0), y(t_0)] - k(t - t_0) \text{ for all } t \ge t_0$$

Taking limit $t \to \infty$ we get

$$\lim_{t\to\infty} E[x(t), y(t)] = -\infty .$$

But this contradicts the hypothesis that E is positive definite in D and the assumption that

$$\lim_{t\to\infty} E[x(t), y(t)] = L > 0$$

So, we must have $L = 0$, that is,

$$\lim_{t\to\infty} E[x(t), y(t)] = 0$$

Since E is positive definite in D, $E(x, y) = 0$ if and only if $(x, y) = (0, 0)$. Thus

$$\lim_{t\to\infty} E[x(t), y(t)] = 0$$

if and only if $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$.

Hence by definition of asymptotic stability of the critical point (0, 0), we see that the critical point (0, 0) of (1) is asymptotically stable.

4.2.6. Example. Consider the system

$$\left.\begin{aligned} \frac{dx}{dt} &= -x + y^2 \\ \frac{dy}{dt} &= -y + x^2 \end{aligned}\right\} \qquad (1)$$

and the function $E(x, y) = x^2 + y^2$. We have seen before in previous example that E is positive definite and E is negative definite (on so negative semi-definite). Hence by above theorem (0, 0) is asymptotically stable critical point of system (1).

4.2.7. Example. Find the nature and stability of the critical point (0, 0) for the non-linear system

$$\left.\begin{aligned}\frac{dx}{dt} &= x+4y-x^2\\ \frac{dy}{dt} &= 6x-y+2xy\end{aligned}\right\} \qquad (1)$$

Also, find the family of the paths of this system.

Solution : Comparing this with

$$\frac{dx}{dt} = ax + by + P_1(x, y)$$

$$\frac{dy}{dt} = cx + dy + Q_1(x, y)$$

We obtain $a = 1$, $b = 4$, $c = 6$, $d = -1$

$$P_1(x, y) = -x^2,\ Q_1(x, y) = 2xy$$

We see that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 6 & -1 \end{vmatrix} = -25 \neq 0$$

and $$\lim_{(x,y)\to(0,0)} \frac{P_1(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{-x^2}{\sqrt{x^2+y^2}} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{Q_1(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}} = 0$$

Hence to investigate the critical point (0, 0) of (1) we consider the linear system

$$\left.\begin{aligned}\frac{dx}{dt} &= x+4y\\ \frac{dy}{dt} &= 6x-y\end{aligned}\right\} \qquad (2)$$

The characteristic equation of (2) is

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

$$\Rightarrow \qquad \lambda^2 - 25 = 0 \Rightarrow \lambda = \pm 5 \qquad \Rightarrow \lambda_1 = 5,\ \lambda_2 = -5$$

Since the roots are real, unequal and of opposite sign, so the critical point (0, 0) is the saddle point of the system (2) and therefore of system (1). Further, we know that saddle point is always unstable so (0, 0) is an unstable critical point of both systems (2) and (1).

Eliminating dt from system (1) we obtain

$$\frac{dy}{dx} = \frac{6x - y + 2xy}{x + 4y - x^2} \qquad (3)$$

which gives the slope of the paths in xy-plane defined by the solutions of (1). Equation (3) is exact and let us find its general solution. Equation (3) can be written as

$$(6x - y + 2xy)dx - (x + 4y - x^2)dy = 0$$

$$M = 6x - y + 2xy,\ N = -(x + 4y - x^2)$$

Its solution is

$$\int_{y \text{ is constant}} M\ dx + \int (\text{Terms of } N \text{ not containing } x)\ dy + C = 0$$

$$\Rightarrow \qquad 3x^2 - yx + x^2y - 2y^2 + C = 0$$

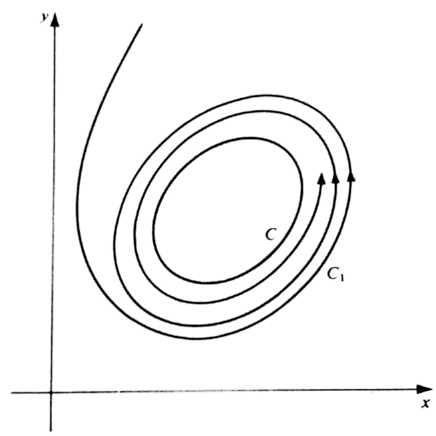
The general solution is

$$3x^2 + x^2y - xy - 2y^2 + C = 0$$

where C is an arbitrary constant. This equation represents the family of paths in xy phase plane.

4.3. Limit Cycle : Consider an autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y) \end{aligned}\right\} \qquad (1)$$



A closed path C of the system (1) which is approached spirally from either the inside or the outside by a non closed path C_1 of the system (1) either $t \to +\infty$ or as $t \to -\infty$ is called a limit cycle of (1).

4.3.1. Example. Consider the autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= y + x(1 - x^2 - y^2) \\ \frac{dy}{dt} &= -x + y(1 - x^2 - y^2) \end{aligned}\right\} \qquad (1)$$

To study this system we shall introduce polar co-ordinates (r, θ) where

$$x = r\cos\theta,\ y = r\sin\theta \qquad (2)$$

from this relation we find that

$$\left.\begin{aligned} x\frac{dx}{dt} + y\frac{dy}{dt} &= r\frac{dr}{dt} \\ x\frac{dy}{dt} - y\frac{dx}{dt} &= r^2\frac{d\theta}{dt} \end{aligned}\right\} \qquad (3)$$

Now multiplying the (1) equation of first by x and second by y and adding we obtain

$$x\frac{dx}{dt} + y\frac{dy}{dt} = (x^2 + y^2)(1 - x^2 - y^2)$$

Introducing the polar coordinates defined by (2) and making use of (3) this become

$$r\frac{dr}{dt} = r^2(1 - r^2)$$

For r $\neq$ 0 we may these write

$$\frac{dr}{dt} = r(1 - r^2)$$

Now multiplying the (1) equation first by y and the second by x and subtracting we obtain

$$y\frac{dx}{dt} - x\frac{dy}{dt} = y^2 + x^2$$

Again using (3) ; we get

$$-r^2\frac{d\theta}{dt} = r^2 \text{ and for } r \neq 0 \text{ we may write it}$$

$$\frac{d\theta}{dt} = -1$$

Thus in polar co-ordinates the system (1) becomes

$$\left.\begin{aligned} \frac{dr}{dt} &= r(1 - r^2) \\ \frac{d\theta}{dt} &= -1 \end{aligned}\right\} \qquad (4)$$

from the second of these equations we find at once that $\theta = -t + t_0$, where t_0 is an arbitrary constant. The first of the equation (4) is separable separating variable we have

$$\frac{dr}{r(1 - r^2)} = dt$$

and an integration using partial fraction yields $\log r^2 - \log(1 - r^2) = 2t + \log|C_0|$

After manipulation we have $r^2 = \dfrac{C_0 e^{2t}}{1 + C_0 e^{2t}}$. Thus we may write $r = \dfrac{1}{\sqrt{1 + Ce^{-2t}}}$ where $C = \dfrac{1}{C_0}$

Thus the solution of system (4) may be written

$$\left.\begin{aligned} r &= \frac{1}{\sqrt{1 + Ce^{-2t}}} \\ \theta &= -t + t_0 \end{aligned}\right\}$$

where c and t_0 are arbitrary constants. We may choose $t_0 = 0$. Then $\theta = -t$, using (2), the solution of the system (1) become

$$x = \frac{\cos t}{\sqrt{1+Ce^{-2t}}} \quad y = \frac{\sin t}{\sqrt{1+Ce^{-2t}}} \qquad (5)$$

The solution (5) of (1) defines the paths of (1) in xy-plane. Explaining their paths for various values of C, we note the following conclusions .

1. If $C = 0$, the path defined by (5) is the circle $x^2 + y^2 = 1$, described in clockwise direction.

2. If $C \neq 0$ the path defined by (5) are not closed paths but rather paths having a spiral behavior.

a) If $C > 0$ the paths are spirals lying inside the circle $x^2 + y^2 = 1$. As $t \to +\infty$, they approach this circle while as $t \to -\infty$, they approach the critical point (0, 0) of (1).

b) If $C < 0$, the paths lie outside the circle $x^2 + y^2 = 1$. These outer paths also this circle as $t \to +\infty$, while as $t \to \log\sqrt{|C|}$ both $|x|$ and $|y|$ becomes infinite.

Since the closed path $x^2 + y^2 = 1$ is approached spirally from both the sides, inside and outside by non-closed paths as $t \to +\infty$. We conclude that this circle is a limit cycle of the system (1).

4.3.2. Bendixson's Non existence criterion.

Hypothesis. Let D be a domain in the xy – plane. Consider the autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x\,,y) \\ \frac{dy}{dt} &= Q(x\,,y) \end{aligned}\right\} \qquad (1)$$

where P and Q are continuous first partial derivatives in D. Suppose that $\frac{\partial P(x\,,\,y)}{\partial x} + \frac{\partial Q(x\,,\,y)}{\partial y}$ has the same sign throughout D.

Conclusion. System (1) has no closed path in domain D

Proof. Let C be a closed curve in D ; let R be the region bounded by C and apply Green's theorem in the plane. We have

$$\int_C \left[P(x\,,\,y)\,dy - Q(x\,,\,y)\,dx\right] = \iint_R \left[\frac{\partial P(x\,,\,y)}{\partial x} + \frac{\partial Q(x\,,\,y)}{\partial y}\right] ds,$$

where the line integral is taken in the positive sense. Now assume that C is a closed path of (1), let $x = f(t)$, $y = g(t)$ be an arbitrary solution of (1) defining C parametrically and let T denote the period of this solution. Then

$$\frac{d\,f(t)}{dt} = P[f(t)\,,\,g(t)]\,,\; \frac{d\,g(t)}{dt} = Q[f(t)\,,\,g(t)]$$

along C and we have

$$\int_C [P(x, y)\, dy - Q(x, y)\, dx] = \int_0^T \left\{ P[f(t), g(t)] \frac{dg(t)}{dt} - Q[f(t), g(t)] \frac{df(t)}{dt} \right\} dt$$

$$= \int_0^T \{ P[f(t), g(t)]\, Q[f(t), g(t)] - Q[f(t), g(t)]\, P[f(t), g(t)] \}\, dt = 0$$

Thus, $\iint_R \left[\frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y} \right] ds = 0$

But this double integral can be zero only if $\frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y}$ changes sign. This is a contradiction, thus C is not a path of (1) and (1) possesses no closed path in d.

4.3.3. Half Path. Let C be a path of the system

$$\left.\frac{dx}{dt} = P(x, y) \;\; and \;\; \frac{dy}{dt} = Q(x, y)\right\} \qquad (1)$$

and let $x = f(t)$, $y = g(t)$ be a solution of (1) defining C. Then we shall call the set of all points of C for $t \geq t_0$, where t_0 is some value of t, a half path of (1). In other words, by a half – path of (1) we mean the set of all points with co – ordinates $[f(t), g(t)]$ for $t_0 \leq t < +\infty$. We denote a half – path of (1) by C^+.

4.3.4. Limit Point of a Half Path. Let C^+ be a half – path of (1) defined by $x = f(t)$, $y = g(t)$ for $t \geq t_0$. Let (x_1, y_1) be a point in the xy – plane. If there exists a sequence of real numbers $\{t_n\}$, n = 1, 2,… such that $t_n \to +\infty$ and $[f(t_n, g(t_n))] \to (x_1, y_1)$ as $n \to +\infty$, then we call (x_1, y_1) a limit point of C^+. The set of all limit points of a half – path C^+ will be called the limit set of C^+ and will be denoted by $L(C^+)$.

4.3.5. Poincare – Bendixson Theorem "Strong" form (Statement only)

Hypothesis. (1) Consider the autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y) \end{aligned}\right\} \qquad (i)$$

Where P and Q have continuous first partial derivatives in a domain D of the xy – plane. Let D, be the bounded sub – domain of D, and let R denote D, plus its boundary.

(2) Let C^+ defined by $x = f(t)$, $y = g(t)$, $t \geq t_0$ be a half – path of (i) contained entirely in R. Suppose the limit set $L(C^+)$ of C^+ contains no critical points of (i).

4.3.6. Index of a critical point. Consider the autonomous system

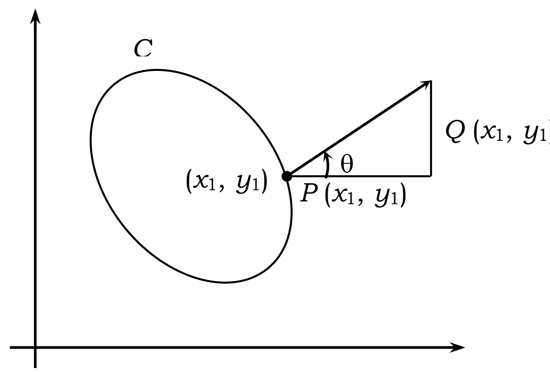
$$\left.\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y) \end{aligned}\right\} \qquad (1)$$

where P and Q have continuous first partial derivatives in a domain D of xy-plane.

Let all the critical points of (1) are isolated. Now consider a simple closed curve C [not necessarily a path of (1)] which passes through no critical point of (1).

Consider a point (x_1, y_1) on C and the vector $[P(x_1, y_1), Q(x_1, y_1)]$ defined by (1) at the point (x_1, y_1).

Let θ denote the angle from the positive x direction to this vector.



Now let (x_1, y_1) describe the curve C once in the anticlockwise direction and return to the original position. As (x_1, y_1) describes the curve C, the vector $[P(x_1, y_1), Q(x_1, y_1)]$ changes continuously, and so the angle θ also varies continuously. When (x_1, y_1) reaches its original position, the angle θ will have changed by an amount $\Delta\theta$. Then we call the number

$$I = \frac{\Delta\theta}{2\pi}$$

the index of the curve C with respect to the system (1).

Remark.

1. Clearly $\Delta\theta$ is either equal to zero or a positive or negative integral multiple of 2π and hence I is either zero or a positive or negative integer.

2. If $[P(x_1, y_1), Q(x_1, y_1)]$ merely oscillates but does not make a complete rotation as (x_1, y_1) describes C, then I is zero.

3. If the net change $\Delta\theta$ in θ is a decrease, then I is negative.

4.4. Floquet Theory. It deals with linear system with periodic coefficients. Consider the linear homogeneous system

$$\frac{dy}{dt} = A(t)\,y, -\infty < t < \infty \qquad (1)$$

where $A(t)$ is $n \times n$ matrix of complex valued continuous functions of real variable t and

$$A(t + w) = A(t) \qquad (2)$$

for some constant $w \neq 0$. Here w is called period of matrix $A(t)$. the system (1) where matrix $A(t)$ is of type (2) is called a periodic system.

Note: The case of periodic variable coefficients can theoretically reduce to the case of constant coefficients. This is the essence of the following theorem.

4.4.1. Theorem. Let $\Phi(t)$ is a fundamental matrix for the system

$$\frac{dy}{dt} = A(t)\,y, -\infty < t < \infty \qquad (1)$$

in which $A(t + w) = A(t)$ (2)

Then $\psi(t) = \Phi(t + w), -\infty < t < \infty$ is also a fundamental matrix for the same system. Moreover, corresponding to every fundamental matrix Φ, there exist a periodic non – singular matrix P with period w and a constant matrix R of order n – by – n such that $\Phi(t) = P(t)\, e^{tk}$.

Proof : As $\Phi(t)$ is a fundamental matrix for the given system, so

$$\Phi'(t) = A(t)\,\Phi(t), -\infty < t < \infty$$

Now, $\psi'(t) = \Phi'(t + w) = A(t + w)\,\Phi(t + w) = A(t)\,\Phi(t + w) = A(t)\,\psi(t), -\infty < t < \infty$

This shows that $\psi(t)$ is also a solution of system (1). Further to show $\psi(t)$ is a fundamental matrix we have to prove that det $\psi(t) \neq 0$ for all t.

Now, det $\Phi(t) \neq 0$, since $\Phi(t)$ is a fundamental matrix.

$\Rightarrow$ det $\Phi(t + w) \neq 0, -\infty < t < \infty \Rightarrow$ det $\psi(t) \neq 0$

Hence $\psi(t)$ is a fundamental matrix for system (1).

Now to prove the second part, we have $\Phi(t)$ and $\Phi(t + w)$ are two fundamental matrix of the system (1). so there exist a constant non – singular matrix c such that

$$\Phi(t + w) = \Phi(t).c \qquad (3)$$

As C is a non – singular matrix, then there exist a constant matrix R such that

$$\log C = w\,R \qquad \text{or} \qquad C = e^{wR} \qquad (4)$$

Putting value of c in equation (3), we get

$$\Phi(t + w) = \Phi(t)\, e^{wR} \qquad (5)$$

Now, we define a matrix P(t) by

$$P(t) = \Phi(t)\, e^{-tR} \qquad (6)$$

Then P(t) is a non – singular matrix as both $\Phi(t)$ and e^{-tR} are non – singular.

Moreover, $P(t + w) = \Phi(t + w)\, e^{-(t+w)R} = \Phi(t)\, e^{wR}\, e^{-(t+w)R} = \Phi(t)\, e^{-tR} = P(t)$

This shows that the matrix P(t) is periodic with period w and from equation (6), we can write

$$\Phi(t) = P(t)\, e^{tR}$$

This completes the proof.

4.5. Sturm Liouville Boundary Value Problems. These problems represents a class of linear boundary value problems. The importance of these problems lies in the fact that they generates set of orthogonal functions. The sets of orthogonal functions are useful in the expansion of a certain class of functions.

The Sturm Liouville differential equation is

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right)+\left(q(t)+\lambda\, r(t)\right)u=0 \qquad (1)$$

where $p(t)$ has continuous derivative, $q(t)$ and $r(t)$ are continuous, and $p(t) > 0$ and $r(t) > 0$ for all t on a real interval [a, b] ; and λ is a parameter whose value is independent of t.

Equation (1) is equivalently written as

$$\mathrm{L}\left(u(t)\right) = -\lambda\, u(t) \qquad (2)$$

where L is Sturm Liouville operator defined by

$$\mathrm{L} = \frac{1}{r(t)}\left[\frac{d}{dt}\left[p(t)\frac{d}{dt}\right]+q(t)\right]$$

Using this, we note that (2) gives

$$\Rightarrow \frac{1}{r(t)}\left[\frac{d}{dt}\left[p(t)\frac{du}{dt}\right]+q(t)u(t)\right] = -\lambda u(t) \Rightarrow \frac{d}{dt}\left[p(t)\frac{du}{dt}\right]+q(t)u(t) = -\lambda\, u(t)\, r(t)$$

$$\Rightarrow \quad \frac{d}{dt}\left(p(t)\frac{du}{dt}\right)+\left[q(t)+\lambda\, r(t)\right]u(t) = 0$$ which is equation (1).

4.5.1. Definition. *A* Sturm Liouville boundary value problem consists of differential equation (1), that is,

$$\frac{d}{dt}\left(p(t)\frac{du}{dt}\right)+\left[q(t)+\lambda\, r(t)\right]u(t) = 0$$ and the boundary conditions

$$\left.\begin{aligned}\alpha_1 u(a)+\alpha_2 u'(a) &= 0\\ \beta_1 u(b)+\beta_2 u'(b) &= 0\end{aligned}\right\} \qquad (*)$$

where α_1, α_2 are constants (not both zero) and β_1 , β_2 are also constants (not both zero).

Remark. Two special cases of Sturm Liouville boundary value problems are those in which supplementary conditions (i.e. boundary conditions) are either of the form $u(\mathrm{a}) = 0$, $u(b) = 0$ or of the form

$$u'(\mathrm{a}) = 0,\ u'(b) = 0$$

4.5.2. Example. Show that the boundary value problem

$$\frac{d^2u}{dt^2} + \lambda u = 0 \qquad (1)$$

with conditions $u(0) = 0$, $u(\pi) = 0$ is a Sturm Liouville problem.

Solution. Given boundary value problem (1) can be written in the form

$$\frac{d}{dt}\left[1.\frac{du}{dt}\right]+\left[0+\lambda.1\right]u = 0$$

and hence (1) is of the form $\frac{d}{dt}\left[p(t)\frac{du}{dt}\right]+\left[q(t)+\lambda\, r(t)\right]u = 0$

where $p(t) = 1$, $q(t) = 0$ and $r(t) = 1$. The supplementary conditions are of the special form

$$u(a) = 0,\ u(b) = 0.$$

4.5.3. Example. Find non-trivial solutions of Sturm Liouville boundary value problem

$$\frac{d^2u}{dt^2}+\lambda u = 0 \quad (1)$$

$$u(0) = 0,\ u(\pi) = 0 \quad (2)$$

Solution. We shall consider separately the three cases $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$.

In each case, we shall first find the general solution of the differential equation (1) and then attempt to determine two arbitrary constants in this general solution so that the supplementary conditions (2) will also be satisfied.

Case I. $\lambda = 0$.

In this case (1) reduces to $\frac{d^2u}{dt^2} = 0$ and so the general solution is

$$u(t) = c_1 + c_2 t \quad (3)$$

We now apply conditions (2) to solution (3). Condition $u(0) = 0$ implies

$$0 = c_1 + c_2.0 \Rightarrow c_1 = 0.$$

and condition $u(\pi) = 0$ implies $0 = c_1 + c_2\pi \Rightarrow c_2 = 0$ [Since $c_1 = 0$].

Thus in order that solution (3) to satisfy conditions (2), we must have $c_1 = c_2 = 0$.

But then the solution (3) becomes $u(t) = 0$ for all t. Thus, in case when the parameter $\lambda = 0$, *t*he only solution of the given problem is the trivial solution.

Case II. $\lambda < 0$.

Differential equation (1) is $\frac{d^2u}{dt^2}+\lambda u = 0$

Its auxiliary equation is $m^2+\lambda = 0$ and $m = \pm\sqrt{-\lambda}$. Since λ is negative, so these roots are real and *u*nequal. Let *us* denote $\sqrt{-\lambda} = \alpha$, we have the general solution

$$u = c_1 e^{\alpha t} + c_2 e^{-\alpha t} \quad (4)$$

We now apply boundary conditions (2) to the equation (4).

Condition $u(0) = 0$ implies $c_1 + c_2 = 0$ (5)

Condition $u(0) = \pi$ implies $c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = 0$ (6)

Clearly $c_1 = c_2 = 0$ is a solution of (5) and (6), but these values of c_1 and c_2 would not give the non – trivial solution of the given problem. We must therefore seek non-zero values of c_1 and c_2 which satisfy (5) and (6).

This system has non-zero solution only if the determinant of coefficients is zero. Therefore, we must have

$$\begin{vmatrix} 1 & 1 \\ e^{\alpha\pi} & e^{-\alpha\pi} \end{vmatrix} = 0$$

$$\Rightarrow \quad e^{-\alpha\pi} - e^{\alpha\pi} = 0 \qquad \Rightarrow \quad e^{-\alpha\pi} = e^{\alpha\pi} \Rightarrow e^{2\alpha\pi} = 1 \Rightarrow \alpha = 0.$$

Since $\alpha = \sqrt{-\lambda}$, we must have $\lambda = 0$. But $\lambda < 0$ in this case. Thus there are no non-trivial solution of the given problem in this case $\lambda < 0$.

Case III. $\lambda > 0$.

In this case *A.E.* is $m^2+\lambda = 0$. So its roots are $m = \pm\sqrt{-\lambda}$

These roots are conjugate complex numbers since $\lambda > 0$. Roots can be written as $\pm\sqrt{\lambda}\, i$. Thus in this case the general solution is of the form

$$u(t) = c_1 \sin\sqrt{\lambda}\, t + c_2 \cos\sqrt{\lambda}\, t \qquad (7)$$

We now apply conditions (2) to this general solution. Condition $u(0) = 0$ implies

$$c_1 \sin 0 + c_2 \cos 0 = 0 \Rightarrow c_2 = 0$$

Condition $u(\pi) = 0$ implies

$$c_1 \sin\sqrt{\lambda}\, \pi + c_2 \cos\sqrt{\lambda}\, \pi = 0$$

$$\Rightarrow \quad c_1 \sin\sqrt{\lambda}\, \pi = 0 \qquad (8)$$

We must therefore satisfy (8).

So we can either set $c_1 = 0$ or $\sin\sqrt{\lambda}\, \pi = 0$. However if $c_1 = 0$ then (since $c_2 = 0$ also) the solution (7) reduces to the trivial solution. Thus to obtain non-trivial solution we can not set $c_1 = 0$ but rather we must set

$$\sin\sqrt{\lambda}\, \pi = 0 \Rightarrow \sqrt{\lambda} = n,\ n = 1, 2, 3, \ldots, \text{ since } \sqrt{\lambda} \text{ is positive}$$

$$\Rightarrow \quad \lambda = n^2,\ n = 1, 2, 3, \ldots$$

Therefore in order that the differential equation (1) have a non-trivial solution of the form (7) satisfying the condition (2), we must have

$$\lambda = n^2 \text{ where } n = 1, 2, 3, \ldots$$

In other words, *t*he parameter λ in (1) must be a number of the infinite sequence 1, 4, 9, 16, …. Also from (7) we see that non-trivial solutions corresponding to $\lambda = n^2$ (n = 1, 2, 3, …) are given by $u(t) = c_n$ sin nt, where c_n is arbitrary non-zero constant.

4.5.4. Characteristic Values and Characteristic Functions.

Consider the Sturm – Liouville problem consisting of the differential equation

$$\frac{d}{dt}\left[p(t)\frac{du}{dt}\right]+\left[q(t)+\lambda\, r(t)\right]u = 0 \qquad (1)$$

and the supplementary conditions

$$\left.\begin{aligned}\alpha_1 u(a) + \alpha_2 u'(a) &= 0\\ \beta_1 u(b) + \beta_2 u'(b) &= 0\end{aligned}\right\} \qquad (2)$$

The values of parameter λ in (1) for which these exist non-trivial solutions of the Sturm Liouville problem are called the characteristic values (or eigen values of the problem). The corresponding non-trivial solutions are called the characteristic functions or the eigen functions of the problem.

4.5.5. Example. Find the characteristic values and the characteristic functions of the Sturm Liouville problem

$$\frac{d^2u}{dt^2}+\lambda u = 0 \qquad (1)$$

$$u(0) = 0,\ u(\pi) = 0,$$

Solution. The values of λ in (1) for which there exist non-trivial solutions of this problem are the values $\lambda = n^2$ where n = 1, 2, 3, …

Then these are the characteristic values of the problem *u*nder consideration. the corresponding non-trivial solutions.

$u(t) = c_n \sin nt$ (n = 1, 2,…),

where c_n is an arbitrary non-zero constants.

4.5.6. Orthogonal Functions. Two functions $u(t)$ and $v(t)$ are said to be orthogonal w.*r.t.* a weight function w(t) on [a, b] iff $\int_a^b u(t)\, v(t)\, w(t)\, dt = 0.$

Let $\{\phi_n(t)\}$ be a sequence of functions on [a, b]. Then these functions are said to be mutually orthogonal w.*r.t.* a weight function w(t) on [a, b] iff $\int_a^b w(t)\, \phi_n(t)\, \phi_m(t)\, dt = 0,\ m \neq n.$

4.5.7. Theorem (Orthogonality of characteristic function).

Consider the Sturm Liouville problem consisting of the differential equation

$$\frac{d}{dt}\left[p(t)\frac{du}{dt}\right]+\left[q(t)+\lambda\, r(t)\right]u = 0 \qquad (1)$$

where p, q, r are real functions s.t. $p(t)$ has a continuous derivative, $q(t)$ and $r(t)$ are continuous, and $p(t) > 0$ and $r(t) > 0$ for all t on a real interval $a \le t \le b$, and λ is a parameter independent of t, and the conditions

$$\left.\begin{aligned} A_1u(a) + A_2u'(a) &= 0 \\ B_1u(b) + B_2u'(b) &= 0 \end{aligned}\right\} \qquad (2)$$

where A_1, A_2, B_1, B_2 are real constants s.t. A_1, A_2 are not both zero and B_1 and B_2 are not both zero.

Let λ_m and λ_n be any two characteristic values of the problem. Let ϕ_m be a characteristic function corresponding to λ_m and let ϕ_n be a characteristic function corresponding to λ_n.

Then characteristic functions ϕ_m and ϕ_n are orthogonal w.r.t. the weight function $r(t)$ on the interval $a \le t \le b$.

Proof. Since ϕ_m is a characteristic function corresponding to λ_m, so the function ϕ_m satisfies the differential equation (1) with $\lambda = \lambda_m$. Similarly, ϕ_n satisfies the equation (1) with $\lambda = \lambda_n$.

Thus we have

$$\frac{d}{dt}\left[p(t)\,\phi'_m(t)\right]+\left[q(t)+\lambda_m\, r(t)\right]\phi_m(t) = 0 \qquad (3)$$

$$\frac{d}{dt}\left[p(t)\,\phi'_n(t)\right]+\left[q(t)+\lambda_n\, r(t)\right]\phi_n(t) = 0 \qquad (4)$$

for all t s.t. $a \le t \le b$.

Multiplying (3) by $\phi_n(t)$ and (4) by $\phi_m(t)$ and substracting, we get

$$\phi_n(t)\frac{d}{dt}\left[p(t)\,\phi'_m(t)\right]+\lambda_m.\phi_m(t)\phi_n(t)\,r(t) - \phi_m(t)\frac{d}{dt}\left[p(t)\,\phi'_n(t)\right]-\lambda_n.\phi_n(t)\phi_m(t)\,r(t) = 0$$

$$\Rightarrow \quad (\lambda_m - \lambda_n)\,\phi_m(t)\,\phi_n(t)\,r(t) = \phi_m(t)\frac{d}{dt}\left[p(t)\,\phi'_n(t)\right] - \phi_n(t)\frac{d}{dt}\left[p(t)\,\phi'_m(t)\right]$$

Integrating it w.r.t. 't' from limit a to b, we get

$$(\lambda_m - \lambda_n)\int_a^b \phi_m(t)\,\phi_n(t)\,r(t)\,dt = \int_a^b \phi_m(t)\frac{d}{dt}\left[p(t)\,\phi'_n(t)\right]dt - \int_a^b \phi_n(t)\frac{d}{dt}\left[p(t)\,\phi'_m(t)\right]dt \qquad (5)$$

Applying integration by parts to each integral in the right member of equation, so this right member becomes

$$\left[\phi_m(t)p(t)\phi'_n(t)\right]_a^b - \left[\phi_n(t)p(t)\phi'_m(t)\right]_a^b = \left[p(t)\left\{\phi_m(t)\phi'_n(t) - \phi_n(t)\phi'_m(t)\right\}\right]_a^b$$

Therefore (5) becomes

$$(\lambda_m - \lambda_n)\int_a^b \phi_m(t)\ \phi_n(t)\ r(t)\ dt = p(b)\left[\phi_m(b)\phi_n'(b) - \phi_n(b)\phi_m'(b)\right] - p(a)\left[\phi_m(a)\phi_n'(a) - \phi_n(a)\phi_m'(a)\right] \qquad (6)$$

Since ϕ_m and ϕ_n are characteristic functions of the problem, *t*hey satisfy the conditions (2) of the problem and now consider the different cases as follows :

Case I. $A_2 = 0$, $B_2 = 0$.

So the conditions reduces to $u(a) = 0$, $u(b) = 0$. Then in this case $\phi_m(a) = \phi_m(b) = 0$

and $\phi_n(a) = \phi_n(b) = 0$ and so equation (6) gives

$$\int_a^b \phi_m(t)\ \phi_n(t)\ r(t)\ dt = 0 \text{ [Since } \lambda_m \neq \lambda_n].$$

Case II. If $A_2 = 0$, $B_2 \neq 0$

So the conditions (2) reduces to $u(a) = 0$ and $\alpha\, u(b) + u'(b) = 0$ where $\alpha = \dfrac{B_1}{B_2}$

Then in this case, we have $\phi_n(a) = 0$. $\phi_m(a) = 0$.

and
$$\left.\begin{aligned} \alpha\ \phi_n(b) + \phi_n'(b) &= 0 \\ \alpha\ \phi_m(b) + \phi_m'(b) &= 0 \end{aligned}\right\} \qquad (7)$$

So the equation (6) becomes

$$(\lambda_m - \lambda_n)\int_a^b \phi_n(t)\ \phi_m(t)\ r(t)\ dt = p(b)\left[\phi_m(b)\phi_n'(b) - \phi_n(b)\phi_m'(b)\right]$$

$$= p(b)\left[\phi_m(b)\phi_n'(b) - \phi_n(b)\phi_m'(b) + \beta\ \phi_m(b)\ \phi_n(b) - \beta\ \phi_m(b)\ \phi_n(b)\right]$$

$$= p(b)\ \left[\phi_m(b)\left\{\alpha\ \phi_n(b) + \phi_n'(b)\right\} - \phi_n(b)\left\{\alpha\ \phi_m(b) + \phi_m'(b)\right\}\right] = 0$$

using (7). Therefore,

$$\int_a^b \phi_m(t)\ \phi_n(t)\ r(t)\ dt = 0 \qquad \text{[Since } \lambda_m \neq \lambda_n]$$

Case III. If $A_2 \neq 0$, $B_2 = 0$

This case is similar to case-(II).

Case IV. If $A_2 \neq 0$, $B_2 \neq 0$. So the conditions are

$$\alpha_1\ u(a) + u'(a) = 0 \text{ where } \alpha_1 = \frac{A_1}{A_2}$$

$$\alpha_2\ u(b) + u'(b) = 0 \text{ where } \alpha_2 = \frac{B_1}{B_2}$$

Then in this case, we have

$$\left.\begin{aligned}\alpha_1\,\phi_m(a) + \phi_m'(a) &= 0\\ \alpha_1\,\phi_n(a) + \phi_n'(a) &= 0\\ \alpha_2\,\phi_m(b) + \phi_m'(b) &= 0\end{aligned}\right\} \quad (8)$$

and

$$\alpha_2\,\phi_n(b) + \phi_n'(b) = 0$$

Now write

$$\phi_m(a)\,\phi_n'(a) - \phi_n(a)\,\phi_m'(a) = \alpha_1\,\phi_n(a)\,\phi_m(a) - \alpha_1\,\phi_n(a)\,\phi_m(a) + \phi_m(a)\,\phi_n'(a) - \phi_n(a)\,\phi_m'(a)$$

$$= \phi_m(a)\,\{\alpha_1\phi_n(a) + \phi_n'(a)\} - \phi_n(a)\{\alpha_1\phi_m(a) + \phi_m'(a)\}$$

$$= \phi_m(a).0 - \phi_n(a).0 = 0 \qquad (9)$$

Similarly, $\phi_m(b)\,\phi_n'(b) - \phi_n(b)\phi_m'(b) = 0$ (10)

Using (9) and (10), equation (6) becomes

$$\int_a^b \phi_n(t)\,\phi_m(t)\,r(t) = 0 \text{ [Since } \lambda_m \neq \lambda_n]$$

So, ϕ_m and ϕ_n are orthogonal w.*r*.*t*. weight function $r(t)$ on a $\le t \le b$.

4.5.8. Theorem. Prove that the eigen values of the Sturm Liouville boundary value problem are always real.

Proof. Consider the Sturm Liouville problem

$$\frac{d}{dt}\left[p(t)\frac{du}{dt}\right] + [q(t) + \lambda\, r(t)]u = 0 \qquad (1)$$

where $p(t)$, $q(t)$ and $r(t)$ are real functions and $p(t)$ has continuous first order derivative and

$$\left.\begin{aligned}A_1u(a) + A_2u'(a) &= 0\\ B_1u(b) + B_2u'(b) &= 0\end{aligned}\right\} \qquad (2)$$

where A_1, A_2, B_1 and B_2 are constant such that $A_1 \neq 0$ and $B_1 \neq 0$

Let λ_n be an eigen value and corresponding eigen function is $\phi_n(t)$. Then we must have

$$\frac{d}{dt}\left[p(t)\frac{d\phi_n}{dt}\right] + [q(t) + \lambda_n\, r(t)]\,\phi_n = 0 \qquad (3)$$

and

$$\left.\begin{aligned}A_1\,\phi_n(a) + A_2\,\phi_n'(a) &= 0\\ B_1\,\phi_n(b) + B_2\,\phi_n'(b) &= 0\end{aligned}\right] \qquad (4)$$

Taking complex conjugate of equation (3) and (4), we get

$$\frac{d}{dt}\left[p(t)\frac{d\overline{\phi}_n}{dt}\right]+\left[q(t)+\overline{\lambda}_n\ r(t)\right]\overline{\phi}_n = 0 \qquad (5)$$

and

$$\left.\begin{array}{l} A_1\ \overline{\phi_n}(a) + A_2\ \overline{\phi_n'}(a) = 0 \\ B_1\ \overline{\phi_n}(b) + B_2\ \overline{\phi_n'}(b) = 0 \end{array}\right] \qquad (6)$$

This show that $\overline{\phi_n}$ is also an eigen function corresponding to an eigen value $\overline{\lambda_n}$. Then by above theorem, we have $(\lambda_n - \overline{\lambda_n})\int_a^b r(t)\ \phi_n(t)\ \overline{\phi_n}(t)\ dt = 0$

$$\Rightarrow\ (\lambda_n - \overline{\lambda_n})\int_a^b r(t)\ |\phi_n(t)|^2 dt = 0$$

Since $r(t) > 0$ and $|\phi_n(t)| \neq 0$, being a non – trivial solution, so we must have

$$\lambda_n - \overline{\lambda_n} = 0 \Rightarrow \lambda_n = \overline{\lambda_n}$$

This shows that λ_n is real.

This completes the proof.

4.5.9. Exercise. Find the characteristic values and corresponding characteristic functions of each of the following Sturm Liouville problems -

(1) $\frac{d^2u}{dt^2} + \lambda u = 0$ $u(0) = 0$, $u\left(\frac{\pi}{2}\right) = 0$

(2) $\frac{d^2u}{dt^2} + \lambda u = 0$ $u(0) = 0$, $u(\mathrm{L}) > 0$, $\mathrm{L} > 0$

(3) $\frac{d}{dt}\left[p(t)\frac{du}{dt}\right] + \frac{\lambda}{t}u = 0$, $u(1) = 0$, $u\left(e^{\pi}\right) = 0$

Answers.

(1) $\lambda = 4n^2$ ($n = 1, 2,\ldots$) $u = c_n \sin 2nt$ ($n = 1, 2,\ldots$)

(2) $\lambda = \left(\frac{n\pi}{L}\right)^2$ ($n = 1, 2,\ldots$), $u = c_n \sin\frac{n\pi t}{L}$ ($n = 1, 2,\ldots$)

(3) $\lambda = n^2$ ($n = 1, 2,\ldots$), $u = c_n \sin(n\log t)$, ($n = 1, 2,\ldots$)

4.6. Check Your Progress.

1. Find all the real critical points of the non – linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= 8x - y^2 \\ \frac{dy}{dt} &= -6y + 6x^2 \end{aligned}\right\}$$

and determine the type and stability of each of these critical points.

2. Find the characteristic values and characteristic functions of the Sturm – Liouville problem

$$\frac{d}{dx}\left[x \ \frac{dy}{dx}\right] + \frac{\lambda}{x}\ y \ = 0$$

$$y'(1) = 0, \ y'\left(e^{2\pi}\right) = 0$$

where we assume that the parameter λ is non-negative.

4.7. Summary.

In this chapter, we discussed about methods to check the stability and asymptotical stability of a critical point and obtained some interesting results which provide the required using Liapunov functions. Also one more important topic of SLBVP is discussed in this chapter which wimm be very useful in further studies, when we deal with Heat, Wave and Laplace equations.

Books Suggested:

1. Ross, S.L., Differential equations, John Wiley and Sons Inc., New York, 1984.
2. Boyce, W.E., Diprima, R.C., Elementary differential equations and boundary value problems, John Wiley and Sons, Inc., New York, 4th edition, 1986.
3. Simmon, G.F., Differential Equations, Tata McGraw Hill, New Delhi, 1993.